

LAPLACIANS ON PERIODIC GRAPHS WITH GUIDES

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ABSTRACT. We consider Laplace operators on periodic discrete graphs perturbed by guides, i.e., graphs which are periodic in some directions and finite in other ones. The spectrum of the Laplacian on the unperturbed graph is a union of a finite number of non-degenerate bands and eigenvalues of infinite multiplicity. We show that the spectrum of the perturbed Laplacian consists of the unperturbed one plus the additional so-called guided spectrum which is a union of a finite number of bands. We estimate the position of the guided bands and their length in terms of geometric parameters of the graph. We also determine the asymptotics of the guided bands for guides with large multiplicity of edges. Moreover, we show that the possible number of guided bands, their length and position can be rather arbitrary for some specific periodic graphs with guides.

1. INTRODUCTION

Laplacians on periodic discrete graphs have attracted a lot of attention due to their applications to the study of electronic properties of real crystalline structures, see, e.g., [H02], [NG04] and the survey [CGPNG09]. However, the arrangement of atoms or molecules in most crystalline materials is not perfect. The regular patterns are interrupted by crystalline defects. These defects are the most important features of the engineering material and are manipulated to control its behavior.

We consider Laplace operators on periodic discrete graphs perturbed by guides (i.e., graphs which are periodic in some directions and finite in others). For example, a guide is a periodic graph embedded into a strip in the case of planar graphs. It is well known that the spectrum of discrete Laplacians on periodic graphs has a band structure with a finite number of flat bands (eigenvalues of infinite multiplicity) [HN09], [HS04], [KS14], [RR07]. The spectrum of the Laplacian on the perturbed graph consists of the spectrum of the Laplacian on the unperturbed periodic graph plus the so-called *guided* spectrum. The additional guided spectrum is a union of a finite number of bands and the corresponding wave-functions are mainly located along the guides. In our paper we study guided spectra of Laplacians. We describe our main goals:

- to estimate the position of guided bands and their lengths in terms of geometric parameters of graphs;
- to determine asymptotics of the guided spectrum for guides with large multiplicity of edges;
- to show that a possible number of guided bands (including flat bands), their length and positions can be rather arbitrary for some specific periodic graphs with guides.

1.1. Discrete Laplacians on periodic graphs. Let $\Gamma = (V, \mathcal{E})$ be a connected infinite graph, possibly having loops and multiple edges, where V is the set of its vertices and \mathcal{E} is the set of its unoriented edges. From the set \mathcal{E} we construct the set \mathcal{A} of oriented edges by considering each edge in \mathcal{E} to have two orientations. An edge starting at a vertex u and ending

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at a vertex v from V will be denoted as the ordered pair $(u, v) \in \mathcal{A}$. Vertices $u, v \in V$ will be called *adjacent* and denoted by $u \sim v$, if $(u, v) \in \mathcal{A}$. We define the degree κ_v of the vertex $v \in V$ as the number of all edges from \mathcal{A} starting at v . We consider graphs with uniformly bounded degrees.

Let $\ell^2(V)$ be the Hilbert space of all functions $f : V \rightarrow \mathbb{C}$ equipped with the norm

$$\|f\|_{\ell^2(V)}^2 = \sum_{v \in V} |f(v)|^2 < \infty.$$

We define the discrete Laplacian (i.e., the combinatorial Laplace operator) Δ on $\ell^2(V)$ by

$$(\Delta f)(v) = \sum_{e=(v,u) \in \mathcal{A}} (f(v) - f(u)), \quad f = (f(v))_{v \in V} \in \ell^2(V), \quad (1.1)$$

where the sum is taken over all oriented edges starting at the vertex $v \in V$. It is well known, see, e.g., [M91], that Δ is self-adjoint and its spectrum satisfies: *the point 0 belongs to the spectrum $\sigma(\Delta)$ containing in $[0, 2\kappa_+]$, i.e.,*

$$0 \in \sigma(\Delta) \subset [0, 2\kappa_+], \quad \text{where} \quad \kappa_+ = \sup_{v \in V} \kappa_v < \infty. \quad (1.2)$$

We consider a $\mathbb{Z}^{\tilde{d}}$ -periodic graph $\Gamma_0 = (V_0, \mathcal{E}_0)$, i.e., a graph satisfying the following conditions:

- 1) Γ_0 is equipped with an action of the free abelian group $\mathbb{Z}^{\tilde{d}}$;
- 2) the quotient graph $\Gamma_* = (V_*, \mathcal{E}_*) = \Gamma_0 / \mathbb{Z}^{\tilde{d}}$ is finite.

We assume that the graphs are embedded into Euclidean space, since in many applications such a natural embedding exists. For example, in the tight-binding approximation real crystalline structures are modeled as discrete graphs embedded into \mathbb{R}^d ($d = 2, 3$) and consisting of vertices (points representing positions of atoms) and edges (representing chemical bonding of atoms), by ignoring the physical characters of atoms and bonds that may be different from one another. But all results of the paper stay valid in the case of abstract periodic graphs (without the assumption of graph embedding into Euclidean space).

For a periodic graph Γ_0 embedded into the space $\mathbb{R}^{\tilde{d}}$, the quotient graph $\Gamma_0 / \mathbb{Z}^{\tilde{d}}$ is a graph on the \tilde{d} -dimensional torus $\mathbb{R}^{\tilde{d}} / \mathbb{Z}^{\tilde{d}}$. Due to the definition, the graph Γ_0 is invariant under translations through vectors $a_1, \dots, a_{\tilde{d}}$ which generate the group $\mathbb{Z}^{\tilde{d}}$:

$$\Gamma_0 + a_s = \Gamma_0, \quad \forall s \in \mathbb{N}_{\tilde{d}}.$$

Here and below for each integer m the set \mathbb{N}_m is given by

$$\mathbb{N}_m = \{1, \dots, m\}. \quad (1.3)$$

We will call the vectors $a_1, \dots, a_{\tilde{d}}$ the *periods of the graph* Γ_0 . In the space $\mathbb{R}^{\tilde{d}}$ we consider a coordinate system with the origin at some point O and with the basis $a_1, \dots, a_{\tilde{d}}$. Below the coordinates of all graph vertices will be expressed in this coordinate system.

Let A be a self-adjoint operator, we denote by $\sigma(A)$, $\sigma_{ac}(A)$, $\sigma_p(A)$, and $\sigma_{fb}(A)$ its spectrum, absolutely continuous spectrum, point spectrum (eigenvalues of finite multiplicity), and the flat band spectrum (eigenvalues of infinite multiplicity), respectively.

We consider the Laplacian defined by (1.1) on the periodic graph Γ_0 as an *unperturbed operator* and denote it by Δ_0 . It is well known that the spectrum $\sigma(\Delta_0)$ of the Laplacian on

periodic graphs is a union of ν spectral bands $\sigma_n(\Delta_0)$:

$$\sigma(\Delta_0) = \bigcup_{n=1}^{\nu} \sigma_n(\Delta_0) = \sigma_{ac}(\Delta_0) \cup \sigma_{fb}(\Delta_0), \quad (1.4)$$

where $\nu = \#V_*$ is the number of vertices of the quotient graph Γ_* , the absolutely continuous spectrum $\sigma_{ac}(\Delta_0)$ consists of non-degenerate bands $\sigma_n(\Delta_0)$. Note that each flat band is a degenerate band. The spectrum $\sigma(\Delta_0)$ is a subset of the interval $[0, \varrho]$:

$$\sigma(\Delta_0) \subset [0, \varrho], \quad \inf \sigma(\Delta_0) = 0, \quad \varrho = \sup \sigma(\Delta_0). \quad (1.5)$$

1.2. Results overview. There are results about spectral properties of the Schrödinger operator $H_0 = \Delta_0 + W$ with a periodic potential W . The decomposition of the operator H_0 into a constant fiber direct integral was obtained in [HN09], [HS04], [RR07] without an exact form of fiber operators and in [KS14], [KS17] with an exact form of fiber operators. In particular, this yields the band-gap structure of the spectrum of the operator H_0 . In [GKT93] the authors described different properties of Schrödinger operators with periodic potentials on the lattice \mathbb{Z}^2 , the simplest \mathbb{Z}^2 -periodic graph. In [LP08], [KS15] the positions of the spectral bands of the Laplacians were estimated in terms of eigenvalues of the operator on finite graphs (the so-called eigenvalue bracketing). The estimate of the total length of all bands $\sigma_n(H_0)$ given by

$$\sum_{n=1}^{\nu} |\sigma_n(H_0)| \leq 2\beta, \quad (1.6)$$

was obtained in [KS14]; where $\beta = \#\mathcal{E}_* + 1 - \nu$ is the so-called Betti number, $\#\mathcal{E}_*$ is the number of edges of the quotient graph Γ_* . Moreover, a global variation of the Lebesgue measure of the spectrum and a global variation of the gap-length in terms of potentials and geometric parameters of the graph were determined. Note that the estimate (1.6) also holds true for magnetic Schrödinger operators with periodic magnetic and electric potentials (see [KS17]). Estimates of the Lebesgue measure of the spectrum of H_0 in terms of eigenvalues of Dirichlet and Neumann operators on a fundamental domain of the periodic graph were described in [KS15]. Estimates of effective masses, associated with the ends of each spectral band, in terms of geometric parameters of the graphs were obtained in [KS16]. Moreover, in the case of the bottom of the spectrum two-sided estimates on the effective mass in terms of geometric parameters of the graphs were determined. The proof of all these results in [KS14]-[KS17] is based on Floquet theory and the exact form of fiber Schrödinger operators from [KS14], [KS17]. The spectra of the discrete Schrödinger operators on graphene nanotubes and nano-ribbons in external fields were discussed in [KK10], [KK10a]. The spectrum of discrete magnetic Laplacians on some planar graphs (the hexagonal lattice, the kagome lattice and so on) was described in [HKR16] and see the references therein.

Discrete Laplacians for some class of periodic graphs with compact perturbations including the square, triangular, diamond, kagome lattices were discussed in [AIM14]. Laplacians on periodic graphs with non-compact perturbations and the stability of their essential spectrum were considered in [SS15]. The spectrum of Laplacians on the lattice \mathbb{Z}^d with pendant edges was studied in [S13]. In the paper [KS16a], the authors considered Schrödinger operators with periodic potentials on periodic discrete graphs perturbed by so-called guided potentials, which are periodic in some directions and finitely supported in others. They described some properties of the additional guided spectrum. We remark that the case of guided potentials

is simpler than the case of periodic graphs with guides and helps us to understand better the properties of the guided spectrum in the case of periodic graphs with guides. It is important that in the case of guided potentials all operators act in the same space. But in the case of periodic graphs with guides this is not true. Note that line defects on the lattice were considered in [C12], [Ku14], [Ku16], [OA12].

Scattering theory for self-adjoint Schrödinger operators with decreasing potentials was investigated in [BS99], [IK12] (for the lattice) and in [PR16] (for periodic graphs). Inverse scattering theory with finitely supported potentials was considered in [IK12] for the case of the lattice \mathbb{Z}^d and in [A12] for the case of the hexagonal lattice. The absence of eigenvalues embedded in the essential spectrum of the operators was discussed in [IM14], [V14]. Trace formulae and global eigenvalues estimates for Schrödinger operators with complex decaying potentials on the lattice were obtained in [KL16]. The Cwikel-Lieb-Rosenblum type bound for the discrete Schrödinger operator on \mathbb{Z}^d was computed in [Ka08], [RS09]. Finally, we note that different properties of Schrödinger operators on graphs were considered in [G15], [Sh98].

2. MAIN RESULTS

2.1. The unperturbed case: periodic graphs. We define the infinite *fundamental graph* \mathcal{C}_0 of the $\mathbb{Z}^{\tilde{d}}$ -periodic graph Γ_0 by

$$\mathcal{C}_0 = (V_0^c, \mathcal{E}_0^c) = \Gamma_0 / \mathbb{Z}^d, \quad V_0^c = V_0 / \mathbb{Z}^d, \quad \mathcal{E}_0^c = \mathcal{E}_0 / \mathbb{Z}^d, \quad d < \tilde{d},$$

where V_0^c is its vertex set and \mathcal{E}_0^c is its set of unoriented edges. Remark that the graph \mathcal{C}_0 is a graph on the cylinder $\mathbb{R}^{\tilde{d}} / \mathbb{Z}^d$ and is $\mathbb{Z}^{\tilde{d}-d}$ -periodic. We also call the fundamental graph \mathcal{C}_0 a *discrete cylinder* or just a *cylinder*. We identify the vertices of the cylinder \mathcal{C}_0 with the vertices of the periodic graph Γ_0 from the strip $\mathcal{S} = [0, 1)^d \times \mathbb{R}^{\tilde{d}-d}$. We will call this infinite vertex set a *fundamental vertex set* of Γ_0 and denote it by the same symbol V_0^c :

$$V_0^c = V_0 \cap \mathcal{S}, \quad \mathcal{S} = [0, 1)^d \times \mathbb{R}^{\tilde{d}-d}. \quad (2.1)$$

Edges of the periodic graph Γ_0 connecting the vertices from the fundamental vertex set V_0^c with the vertices from $V_0 \setminus V_0^c$ will be called *bridges*. Bridges always exist and provide the connectivity of the periodic graph. The set of all bridges of the graph Γ_0 we denote by \mathcal{B} .

2.2. The perturbed case: periodic graphs with guides. We define the *union* of two graphs $G_0 = (\mathcal{V}_0, \mathcal{E}_0)$ and $G_1 = (\mathcal{V}_1, \mathcal{E}_1)$ is a graph G given by

$$G = G_0 \cup G_1 = (\tilde{V}, \tilde{\mathcal{E}}) \quad \tilde{V} = \mathcal{V}_0 \cup \mathcal{V}_1, \quad \tilde{\mathcal{E}} = \mathcal{E}_0 \cup \mathcal{E}_1.$$

Now we define a periodic graph with guides. Let $\Gamma_1 = (V_1, \mathcal{E}_1)$ be a finite graph, possibly not connected, such that all vertices of Γ_1 are contained in the strip \mathcal{S} and the graph $\Gamma_0 \cup \Gamma_1$ is connected. A \mathbb{Z}^d -periodic graph

$$\Gamma_1^g = \bigcup_{m \in \mathbb{Z}^d} (\Gamma_1 + m) \quad (2.2)$$

will be called a *guide* with the fundamental graph Γ_1 . We define a perturbed graph Γ as a union of the unperturbed periodic graph Γ_0 and the perturbation Γ_1^g :

$$\Gamma = \Gamma_0 \cup \Gamma_1^g. \quad (2.3)$$

We will call the graph Γ a *periodic graph with a guide* Γ_1^g or a *perturbed graph*.

Due to the definition (2.3) of the perturbed graph Γ , the *perturbed* cylinder $\mathcal{C} = \Gamma/\mathbb{Z}^d = (V^c, \mathcal{E}^c)$ for Γ is a union of the unperturbed cylinder $\mathcal{C}_0 = (V_0^c, \mathcal{E}_0^c)$ and the finite graph $\Gamma_1 = (V_1, \mathcal{E}_1)$:

$$\mathcal{C} = \mathcal{C}_0 \cup \Gamma_1, \quad (2.4)$$

i.e.,

$$V^c = V_0^c \cup V_1, \quad \mathcal{E}^c = \mathcal{E}_0^c \cup \mathcal{E}_1, \quad \mathcal{A}^c = \mathcal{A}_0^c \cup \mathcal{A}_1, \quad (2.5)$$

where \mathcal{A}^c , \mathcal{A}_0^c and \mathcal{A}_1 are the sets of all doubled oriented edges of \mathcal{C} , \mathcal{C}_0 and Γ_1 , respectively.

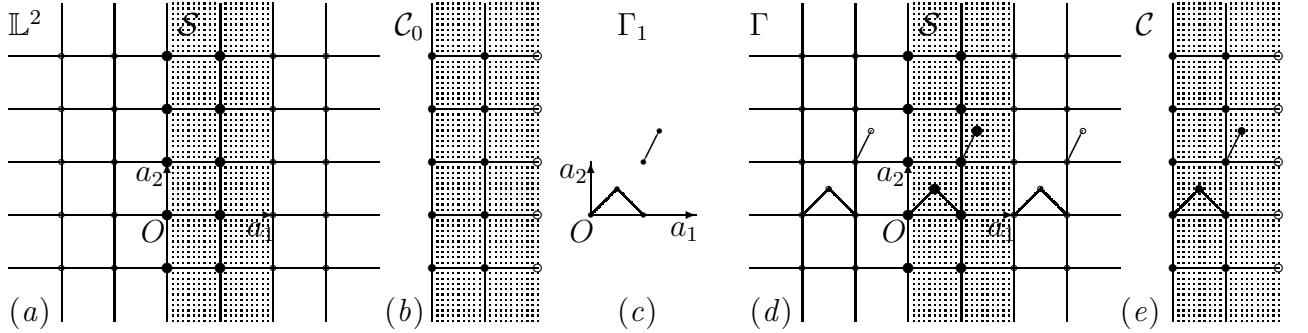


FIGURE 1. *a)* The square lattice \mathbb{L}^2 ; the vertices from the set V_0^c are big black points; the strip \mathcal{S} is shaded; *b)* the unperturbed cylinder $\mathcal{C}_0 = \mathbb{L}^2/\mathbb{Z}$ (the edges of the strip are identified); *c)* a perturbation Γ_1 with two connected components; *d)* the perturbed square lattice $\Gamma = \mathbb{L}^2 \cup \Gamma_1^g$; *e)* the perturbed cylinder $\mathcal{C} = \Gamma/\mathbb{Z}$ (the edges of the strip are identified).

Example. For the square lattice \mathbb{L}^2 with the periods a_1, a_2 , see Fig.1.a, the unperturbed cylinder $\mathcal{C}_0 = \mathbb{L}^2/\mathbb{Z} = (V_0^c, \mathcal{E}_0^c)$ is shown in Fig.1.b. The vertices from the set V_0^c are big black points in Fig.1.a. A perturbation Γ_1 , the perturbed square lattice $\Gamma = \mathbb{L}^2 \cup \Gamma_1^g$ and the perturbed cylinder $\mathcal{C} = \Gamma/\mathbb{Z}$ are shown in Fig.1.c,d,e.

2.3. Floquet decomposition and the spectrum of perturbed Laplacians. We describe the basic spectral properties of Laplacians on periodic graphs with guides.

Proposition 2.1. *i) The Laplacian Δ on a perturbed graph Γ has the following decomposition into a constant fiber direct integral for some unitary operator $U : \ell^2(V) \rightarrow \mathcal{H}$:*

$$\mathcal{H} = \int_{\mathbb{T}^d}^{\oplus} \ell^2(V^c) \frac{d\vartheta}{(2\pi)^d}, \quad U\Delta U^{-1} = \int_{\mathbb{T}^d}^{\oplus} \Delta(\vartheta) \frac{d\vartheta}{(2\pi)^d}, \quad (2.6)$$

where $\mathbb{T}^d = \mathbb{R}^d/(2\pi\mathbb{Z})^d$ and the fiber Laplacian $\Delta(\vartheta)$ on the fiber space $\ell^2(V^c)$ is given by

$$(\Delta(\vartheta)f)(v) = \sum_{\mathbf{e}=(v,u) \in \mathcal{A}^c} (f(v) - e^{i\langle \tau(\mathbf{e}), \vartheta \rangle} f(u)), \quad v \in V^c, \quad f \in \ell^2(V^c). \quad (2.7)$$

Here $\tau(\mathbf{e}) \in \mathbb{Z}^d$ is the index of the edge $\mathbf{e} \in \mathcal{A}^c$ defined by (3.3), (3.4), V^c and \mathcal{A}^c are the vertex set and the set of oriented edges of the cylinder \mathcal{C} , respectively; $\langle \cdot, \cdot \rangle$ is the inner product in \mathbb{R}^d .

ii) For each $\vartheta \in \mathbb{T}^d$ the spectrum of the fiber operator $\Delta(\vartheta)$ has the form

$$\sigma(\Delta(\vartheta)) = \sigma_{ac}(\Delta(\vartheta)) \cup \sigma_{fb}(\Delta(\vartheta)) \cup \sigma_p(\Delta(\vartheta)), \quad (2.8)$$

$$\sigma_{ac}(\Delta(\vartheta)) = \sigma_{ac}(\Delta_0(\vartheta)), \quad \sigma_{fb}(\Delta(\vartheta)) = \sigma_{fb}(\Delta_0(\vartheta)), \quad (2.9)$$

where $\Delta_0(\vartheta)$ is the fiber operator for the unperturbed Laplacian Δ_0 on the periodic graph Γ_0 , $\sigma_p(\Delta(\vartheta))$ is the set of all eigenvalues of $\Delta(\vartheta)$ of finite multiplicity given by

$$\lambda_{N_\vartheta}(\vartheta) \leq \dots \leq \lambda_2(\vartheta) \leq \lambda_1(\vartheta), \quad N_\vartheta \leq p := \text{rank } \Delta_1 = \nu_1 - c_{\Gamma_1}, \quad \nu_1 = \#V_1, \quad (2.10)$$

Δ_1 is the Laplacian on the finite graph $\Gamma_1 = (V_1, \mathcal{E}_1)$ and c_{Γ_1} is the number of connected components of Γ_1 , $\#A$ denotes the number of all elements of the set A .

Remark. The fiber Laplacian $\Delta(\vartheta)$, $\vartheta \in \mathbb{T}^d$, can be considered as a magnetic Laplacian on the cylinder \mathcal{C} (see [HS99], [KS17]).

Proposition 2.1 and standard arguments (see Theorem XIII.85 in [RS78]) describe the spectrum of the Laplacian Δ on a perturbed graph Γ . Since $\Delta(\vartheta)$ is self-adjoint and real analytic on the torus $\mathbb{T}^d = \mathbb{R}^d / (2\pi\mathbb{Z})^d$, each $\lambda_j(\cdot)$ is a real and piecewise analytic function on \mathbb{T}^d and creates the *guided band* $\mathfrak{s}_j(\Delta)$ given by

$$\mathfrak{s}_j(\Delta) = [\lambda_j^-, \lambda_j^+] = \lambda_j(\mathbb{T}^d), \quad j = 1, \dots, N, \quad N = \max_{\vartheta \in \mathbb{T}^d} N_\vartheta \leq p. \quad (2.11)$$

Thus, the spectrum of the Laplacian Δ on the perturbed graph Γ has the form

$$\sigma(\Delta) = \bigcup_{\vartheta \in \mathbb{T}^d} \sigma(\Delta(\vartheta)) = \sigma(\Delta_0) \cup \mathfrak{s}(\Delta),$$

where $\sigma(\Delta_0)$ is defined by (1.4) and

$$\mathfrak{s}(\Delta) = \bigcup_{\vartheta \in \mathbb{T}^d} \sigma_p(\Delta(\vartheta)) = \bigcup_{j=1}^N \mathfrak{s}_j(\Delta) = \mathfrak{s}_{ac}(\Delta) \cup \mathfrak{s}_{fb}(\Delta), \quad (2.12)$$

$\mathfrak{s}_{ac}(\Delta)$ and $\mathfrak{s}_{fb}(\Delta)$ are the absolutely continuous part and the flat band part of the guided spectrum $\mathfrak{s}(\Delta)$, respectively. An open interval between two neighboring non-degenerate bands is called a *spectral gap*. The guided spectrum $\mathfrak{s}(\Delta)$ may partly lie above the spectrum of the unperturbed operator Δ_0 , on the spectrum of Δ_0 and in the gaps of Δ_0 .

We formulate simple sufficient conditions for the existence of guided flat bands of the perturbed Laplacian Δ .

Proposition 2.2. *Let Γ_0 be a periodic graph. Assume that ζ is an eigenvalue of the Laplacian Δ_1 on a finite graph Γ_1 with an eigenfunction $f \in \ell^2(V_1)$ equal to zero on $V_0^c \cap V_1$, i.e.,*

$$f(v) = 0, \quad \forall v \in V_{01} = V_0^c \cap V_1. \quad (2.13)$$

Then $\{\zeta\}$ is a guided flat band of the Laplacian Δ on the perturbed graph $\Gamma = \Gamma_0 \cup \Gamma_1^g$.

Example. We consider the perturbed square lattice $\Gamma = \mathbb{L}^2 \cup \Gamma_1^g$ shown in Fig.2.a. For each $\vartheta \in \mathbb{T} = (-\pi, \pi]$ the spectrum of the fiber Laplacian $\Delta(\vartheta)$ has the form

$$\sigma(\Delta(\vartheta)) = \sigma_{ac}(\Delta(\vartheta)) \cup \sigma_p(\Delta(\vartheta)), \quad \sigma_{ac}(\Delta(\vartheta)) = \sigma(\Delta_0(\vartheta)) = [2 - 2\cos\vartheta, 6 - 2\cos\vartheta],$$

$\sigma_p(\Delta(\vartheta))$ consists of two eigenvalues $\lambda_1(\vartheta)$ and $\lambda_2(\vartheta) = 3$, see Fig.2.d.

The spectrum of the Laplacian Δ on Γ has the form

$$\sigma(\Delta) = \sigma(\Delta_0) \cup \mathfrak{s}(\Delta), \quad \sigma(\Delta_0) = [0, 8],$$

where the guided spectrum $\mathfrak{s}(\Delta)$ is given by (see Fig.2.d and details in Proposition 5.3)

$$\begin{aligned} \mathfrak{s}(\Delta) &= \mathfrak{s}_{ac}(\Delta) \cup \mathfrak{s}_{fb}(\Delta), & \mathfrak{s}_{ac}(\Delta) &= \mathfrak{s}_1(\Delta) = \lambda_1(\mathbb{T}) \approx [10.6; 13.9], \\ & & \mathfrak{s}_{fb}(\Delta) &= \mathfrak{s}_2(\Delta) = \lambda_2(\mathbb{T}) = \{3\}. \end{aligned}$$

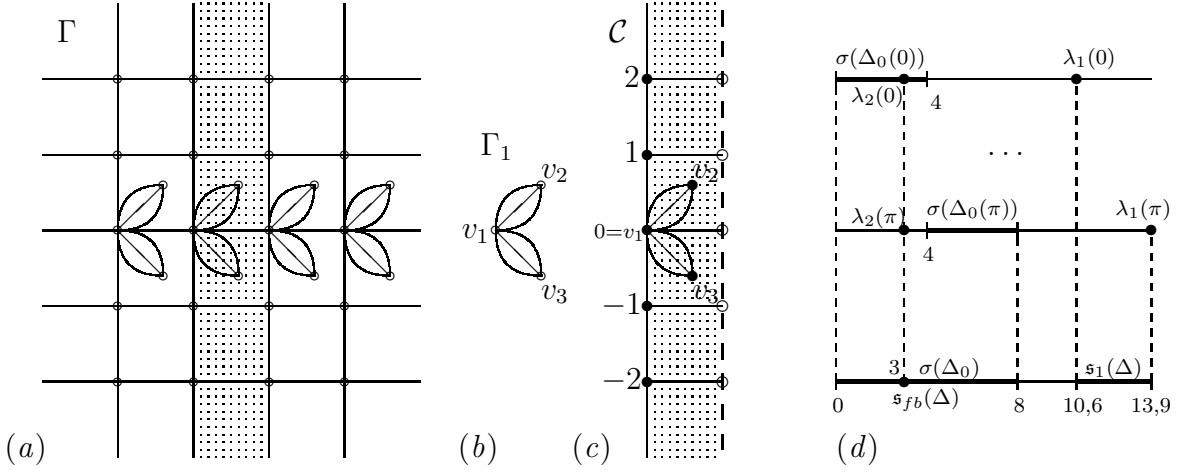


FIGURE 2. *a)* The perturbed square lattice $\Gamma = \mathbb{Z}^2 \cup \Gamma_1^g$; *b)* The finite graph Γ_1 ; *c)* the perturbed cylinder $\mathcal{C} = \Gamma/\mathbb{Z}$; *d)* the spectra of the fiber Laplacians $\Delta(\vartheta)$ as $\vartheta = 0, \pi$ and the Laplacian Δ .

The Laplacian Δ_1 on the finite graph Γ_1 has the eigenvalue $\lambda = 3$ with an eigenfunction f such that $f(0) = 0$. Then, due to Proposition 2.2, $\{3\}$ is a guided flat band of Δ on the perturbed square lattice $\Gamma = \mathbb{Z}^2 \cup \Gamma_1^g$. Here we remark that we do not know an example of a Schrödinger operator with a guided potential on a periodic graph having a guided flat band.

2.4. Estimates of guided bands. We consider the guided bands from (2.11) (or their parts) above the spectrum of the unperturbed Laplacian Δ_0 :

$$\mathfrak{s}_j^o(\Delta) = \mathfrak{s}_j(\Delta) \cap [\varrho, +\infty) \neq \emptyset, \quad j = 1, \dots, N_g, \quad N_g \leq N, \quad (2.14)$$

recall that $\varrho = \sup \sigma(\Delta_0)$. The Laplacian Δ_1 on the finite graph $\Gamma_1 = (V_1, \mathcal{E}_1)$ has the eigenvalue 0 of multiplicity c_{Γ_1} and p positive eigenvalues ζ_j labeled by

$$0 < \zeta_p \leq \dots \leq \zeta_2 \leq \zeta_1, \quad p = \nu_1 - c_{\Gamma_1}, \quad \nu_1 = \#V_1, \quad (2.15)$$

counting multiplicity, where c_{Γ_1} is the number of connected components of the graph Γ_1 .

Proposition 2.1 and the standard perturbation theory give the estimates of the position of the bands $\mathfrak{s}_j^o(\Delta)$ and their number N_g by (for more details see Corollary 4.1)

$$\mathfrak{s}_j^o(\Delta) \subset [\zeta_j, \zeta_j + \varrho], \quad N_g \geq \#\{j \in \mathbb{N}_p : \zeta_j > \varrho\}, \quad (2.16)$$

where $\#A$ is the number of elements of the set A . In particular, this yields that if the eigenvalues of Δ_1 satisfy $\zeta_p > \varrho$ and $\zeta_j - \zeta_{j+1} > \varrho$ for all $j \in \mathbb{N}_{p-1}$, then the guided spectrum of the Laplacian Δ consists of exactly p guided bands separated by gaps.

In order to formulate our main result we define the set $\mathcal{B}^c = \mathcal{B}/\mathbb{Z}^d$ of all bridges of the cylinder $\mathcal{C} = (V^c, \mathcal{E}^c)$ and the modified cylinder $\mathcal{C}^m = (V^c, \mathcal{E}^c \setminus \mathcal{B}^c)$, which is obtained from \mathcal{C} by deleting all its bridges. We consider the Laplacian Δ^m defined by (1.1) on the modified cylinder \mathcal{C}^m . This Laplacian Δ^m has at most $p = \text{rank } \Delta_1$ eigenvalues $\tilde{\mu}_1 \geq \tilde{\mu}_2 \geq \dots$. Define μ_j by

$$\mu_j = \max\{\tilde{\mu}_j, \sup \sigma_{ess}(\Delta^m)\}, \quad j = 1, 2, \dots, p. \quad (2.17)$$

We estimate the position of the guided bands $\mathfrak{s}_j^o(\Delta)$ defined by (2.14) in terms of the eigenvalues of the operator Δ^m and the number of bridges on the cylinder \mathcal{C} .

Theorem 2.3. *i) Let Δ be the Laplacian on a perturbed graph Γ and let μ_j be defined by (2.17). Then each guided band $\mathfrak{s}_j^o(\Delta)$, $j = 1, \dots, N_g$, defined by (2.14) satisfies*

$$\mathfrak{s}_j^o(\Delta) \subset [\mu_j, \mu_j + 2\beta_+], \quad \beta_+ = \max_{v \in V^c} \beta_v, \quad (2.18)$$

where β_v is the number of bridges on \mathcal{C} starting at the vertex $v \in V^c$.

ii) Moreover, for any $\varepsilon > 0$ there exists a perturbed graph Γ such that each non-degenerate guided band length $|\mathfrak{s}_j^o(\Delta)| > 2\beta_+ - \varepsilon$, $j = 1, \dots, N_g$.

Remark. For most of graphs the number $\beta_+ = 1$, then the guided band length $|\mathfrak{s}_j^o(\Delta)| \leq 2$ for all $j = 1, \dots, N_g$, but for specific graphs β_+ may be any given integer number.

Let $\Gamma_t = (V_1, \mathcal{E}_t)$ be a finite graph obtained from the graph $\Gamma_1 = (V_1, \mathcal{E}_1)$ considering each edge of Γ_1 to have the multiplicity $t \in \mathbb{N}$. We consider the Laplacian Δ_t acting on a perturbed graph $\Gamma = \Gamma_0 \cup \Gamma_t^g$ and discuss the guided spectrum of Δ_t for large t .

Theorem 2.4. *Let Δ_t be the Laplacian on the perturbed graph $\Gamma = \Gamma_0 \cup \Gamma_t^g$, where Γ_0 is any periodic graph and $t \in \mathbb{N}$ is large enough. Then the guided spectrum of the Laplacian Δ_t consists of exactly p guided bands separated by gaps, where p is defined in (2.15), and the following statements hold true:*

i) Let ζ_j for some $j \in \mathbb{N}_p$ be a simple positive eigenvalue of the Laplacian Δ_1 on the graph Γ_1 with a normalized eigenfunction $f_j \in \ell^2(V_1)$.

- If $f_j = 0$ on $V_{01} = V_0^c \cap V_1$, then $\{t\zeta_j\}$ is a guided flat band of the Laplacian Δ_t .*
- If $f_j \neq 0$ on V_{01} , then the guided band $\mathfrak{s}_j(\Delta_t) = [\lambda_j^-(t), \lambda_j^+(t)]$ satisfies*

$$\begin{aligned} \lambda_j^\pm(t) &= t\zeta_j + W_j^\pm + O(1/t), \\ |\mathfrak{s}_j(\Delta_t)| &= W_j^\bullet + O(1/t), \end{aligned} \quad (2.19)$$

as $t \rightarrow \infty$, where

$$\begin{aligned} W_j^- &= \min_{\vartheta \in \mathbb{T}^d} W_j(\vartheta), & W_j^+ &= \max_{\vartheta \in \mathbb{T}^d} W_j(\vartheta), \\ W_j^\bullet &= W_j^+ - W_j^-, & W_j^\bullet &\leq 2\beta_{01}, \end{aligned} \quad (2.20)$$

for some function W_j defined by the formula (4.13). Here β_{01} is the number of all oriented bridges connecting the vertices from V_{01} on the cylinder \mathcal{C} .

ii) In particular, if the set V_{01} consists of one vertex v , then

$$f_j^2(v)\beta_{01} \leq W_j^\bullet \leq 2f_j^2(v)\beta_{01}. \quad (2.21)$$

Moreover, $W_j^\bullet = 0$ iff $\beta_{01} = 0$.

iii) Let all positive eigenvalues ζ_j , $j \in \mathbb{N}_p$, of Δ_1 be distinct. Then the Lebesgue measure $|\mathfrak{s}(\Delta_t)|$ of the guided spectrum of the Laplacian Δ_t satisfies

$$|\mathfrak{s}(\Delta_t)| = \sum_{j=1}^p W_j^\bullet + O(1/t). \quad (2.22)$$

iv) In particular, if there is no bridge connecting the vertices from the set V_{01} on the cylinder \mathcal{C} , then $W_j^\bullet = 0$ for each $j \in \mathbb{N}_p$ and the second identity in (2.19) and the formula (2.22) take the form $|\mathfrak{s}_j(\Delta_t)| = O(1/t)$, $j \in \mathbb{N}_p$, and $|\mathfrak{s}(\Delta_t)| = O(1/t)$, respectively.

Now we describe geometric properties of the guided spectrum for periodic graphs with specific guides.

Corollary 2.5. *Let Γ_0 be a periodic graph with an unperturbed cylinder $\mathcal{C}_0 = (V_0^c, \mathcal{E}_0^c)$. Then the following statements hold true.*

- i) *For any constant $C > 0$ there exists a finite graph Γ_t , $t \in \mathbb{N}$, such that the Lebesgue measure of the guided spectrum $\mathfrak{s}(\Delta)$ of the perturbed Laplacian Δ on $\Gamma = \Gamma_0 \cup \Gamma_t^g$ satisfies $|\mathfrak{s}(\Delta)| > C$ and all guided bands are non-degenerate.*
- ii) *Let, in addition, there exist a vertex $v \in V_0^c$ such that there is no bridge on \mathcal{C}_0 starting at v . Then for any small $\varepsilon > 0$ there exists a finite graph Γ_t such that the Lebesgue measure of the guided spectrum $\mathfrak{s}(\Delta)$ of the perturbed Laplacian Δ on $\Gamma = \Gamma_0 \cup \Gamma_t^g$ satisfies $|\mathfrak{s}(\Delta)| < \varepsilon$.*
- iii) *For any constant $\lambda_0 > 0$ there exists a finite graph Γ_t such that the guided spectrum $\mathfrak{s}(\Delta)$ of the perturbed Laplacian Δ on $\Gamma = \Gamma_0 \cup \Gamma_t^g$ satisfies $\mathfrak{s}(\Delta) \cap (\lambda_0, +\infty) \neq \emptyset$.*
- iv) *There exists a finite graph Γ_1 such that the guided spectrum $\mathfrak{s}(\Delta)$ of the perturbed Laplacian Δ on $\Gamma = \Gamma_0 \cup \Gamma_1^g$ has a degenerate guided band.*

Thus, roughly speaking, the guided spectrum can be any set above the unperturbed spectrum. Its Lebesgue measure can be arbitrarily large or arbitrarily small.

We present the plan of our paper. In Section 3 we introduce the notion of edge indices and prove Proposition 2.1 about the decomposition of the Laplacian on periodic graphs with guides into a constant fiber direct integral. In Section 4 we prove Theorems 2.3, 2.4 and Corollary 2.5. Section 5 is devoted to properties of the guided spectrum for the square lattice with specific guides.

3. DIRECT INTEGRAL FOR LAPLACIANS ON PERIODIC GRAPHS WITH GUIDES

3.1. Edge indices. In order to give a decomposition of Laplacians on periodic graphs with guides into a constant fiber direct integral with a precise representation of fiber operators we need to define *an edge index*. Recall that an edge index was introduced in [KS14] and it was important to study the spectrum of Laplacians and Schrödinger operators on periodic graphs, since fiber operators are expressed in terms of edge indices (see (2.7)).

For any $v \in V$ the following unique representation holds true:

$$v = v_0 + [v], \quad v_0 \in V^c, \quad [v] \in \mathbb{Z}^d, \quad (3.1)$$

where V^c is the fundamental vertex set of the graph $\Gamma = (V, \mathcal{E})$ defined by

$$V^c = V \cap \mathcal{S}, \quad \mathcal{S} = [0, 1)^d \times \mathbb{R}^{\tilde{d}-d}. \quad (3.2)$$

In other words, each vertex v can be obtained from a vertex $v_0 \in V^c$ by the shift by a vector $[v] \in \mathbb{Z}^d$. For any oriented edge $\mathbf{e} = (u, v) \in \mathcal{A}$ we define **the edge "index"** $\tau(\mathbf{e})$ as the integer vector given by

$$\tau(\mathbf{e}) = [v] - [u] \in \mathbb{Z}^d, \quad (3.3)$$

where, due to (3.1), we have

$$u = u_0 + [u], \quad v = v_0 + [v], \quad u_0, v_0 \in V^c, \quad [u], [v] \in \mathbb{Z}^d.$$

We note that edges connecting vertices from the fundamental vertex set V^c have zero indices.

We define a surjection $\mathfrak{f}_{\mathcal{A}} : \mathcal{A} \rightarrow \mathcal{A}^c = \mathcal{A}/\mathbb{Z}^d$, which map each edge to its equivalence class. If \mathbf{e} is an oriented edge of the graph Γ , then there is an oriented edge $\mathbf{e}_* = \mathfrak{f}_{\mathcal{A}}(\mathbf{e})$ on the cylinder $\mathcal{C} = \Gamma/\mathbb{Z}^d$. For the edge $\mathbf{e}_* \in \mathcal{A}^c$ we define the edge index $\tau(\mathbf{e}_*)$ by

$$\tau(\mathbf{e}_*) = \tau(\mathbf{e}). \quad (3.4)$$

In other words, edge indices of the cylinder \mathcal{C} are induced by edge indices of the graph Γ . Edges with nonzero indices are called *bridges*. Edge indices, generally speaking, depend on the choice of the coordinate origin O and the periods $a_1, \dots, a_{\tilde{d}}$ of the graph Γ . But in a fixed coordinate system indices of the cylinder edges are uniquely determined by (3.4), since

$$\tau(\mathbf{e} + m) = \tau(\mathbf{e}), \quad \forall (\mathbf{e}, m) \in \mathcal{A} \times \mathbb{Z}^d.$$

We note that, due to the definition of periodic graphs with guides,

$$\tau(\mathbf{e}) = 0, \quad \forall \mathbf{e} \in \mathcal{A}_1. \quad (3.5)$$

3.2. Direct integrals. We prove Proposition 2.1 about the decomposition of Laplacians on periodic graphs with guides into a constant fiber direct integral.

Proof of Proposition 2.1.i) Repeating the arguments from the proof of Theorem 1.1 in [KS14] we obtain (2.6), (2.7), where the unitary operator $U : \ell^2(V) \rightarrow \mathcal{H}$ has the form

$$(Uf)(\vartheta, v) = \sum_{m \in \mathbb{Z}^d} e^{-i\langle m, \vartheta \rangle} f(v + m), \quad (\vartheta, v) \in \mathbb{T}^d \times V^c, \quad f \in \ell^2(V). \quad (3.6)$$

The Hilbert space \mathcal{H} defined in (2.6) is equipped with the norm $\|g\|_{\mathcal{H}}^2 = \int_{\mathbb{T}^d} \|g(\vartheta, \cdot)\|_{\ell^2(V^c)}^2 \frac{d\vartheta}{(2\pi)^d}$, where the function $g(\vartheta, \cdot) \in \ell^2(V^c)$ for almost all $\vartheta \in \mathbb{T}^d$. ■

In order to prove the next item of Proposition 2.1 we need the following lemma.

Lemma 3.1. *Let P and P_1 be the orthogonal projections of $\ell^2(V^c)$ onto the subspaces $\ell^2(V_0^c)$ and $\ell^2(V_1)$, respectively. Then each fiber Laplacian $\Delta(\vartheta)$, $\vartheta \in \mathbb{T}^d$, defined by (2.7) has the following decomposition:*

$$\Delta(\vartheta) = P\Delta_0(\vartheta)P + P_1\Delta_1P_1, \quad (3.7)$$

where $\Delta_0(\vartheta)$ is the fiber operator for the unperturbed Laplacian Δ_0 on the periodic graph Γ_0 , Δ_1 is the Laplacian on the finite graph $\Gamma_1 = (V_1, \mathcal{E}_1)$.

Proof. The Laplacian Δ_0 on the unperturbed periodic graph $\Gamma_0 = (V_0, \mathcal{E}_0)$ has a decomposition into a constant fiber direct integral for some unitary operator $U_0 : \ell^2(V_0) \rightarrow \mathcal{H}_0$:

$$\mathcal{H}_0 = \int_{\mathbb{T}^d}^{\oplus} \ell^2(V_0^c) \frac{d\vartheta}{(2\pi)^d}, \quad U_0\Delta_0U_0^{-1} = \int_{\mathbb{T}^d}^{\oplus} \Delta_0(\vartheta) \frac{d\vartheta}{(2\pi)^d}, \quad (3.8)$$

where the fiber Laplacian $\Delta_0(\vartheta)$ acts on the fiber space $\ell^2(V_0^c)$ and is given by

$$(\Delta_0(\vartheta)f_0)(v) = \sum_{\mathbf{e}=(v,u) \in \mathcal{A}_0^c} (f_0(v) - e^{i\langle \tau(\mathbf{e}), \vartheta \rangle} f_0(u)), \quad v \in V_0^c, \quad f_0 \in \ell^2(V_0^c). \quad (3.9)$$

For each $f \in \ell^2(V^c)$ we have

$$\begin{aligned} & \langle (P\Delta_0(\vartheta)P + P_1\Delta_1P_1)f, f \rangle_{V^c} = \langle \Delta_0(\vartheta)Pf, Pf \rangle_{V_0^c} + \langle \Delta_1P_1f, P_1f \rangle_{V_1} \\ & = \sum_{v \in V_0^c} (\Delta_0(\vartheta)Pf)(v) \bar{f}(v) + \sum_{v \in V_1} (\Delta_1P_1f)(v) \bar{f}(v) = \sum_{v \in V_0^c \setminus V_{01}} (\Delta_0(\vartheta)Pf)(v) \bar{f}(v) \\ & + \sum_{v \in V_1 \setminus V_{01}} (\Delta_1P_1f)(v) \bar{f}(v) + \sum_{v \in V_{01}} ((\Delta_0(\vartheta)P + \Delta_1P_1)f)(v) \bar{f}(v), \end{aligned}$$

where $\langle \cdot, \cdot \rangle_V$ denotes the inner product in $\ell^2(V)$. Substituting the definitions (1.1) and (3.9) of the Laplacian Δ_1 and the fiber Laplacian $\Delta_0(\vartheta)$ into this formula and using the identities (2.5), (3.5) and (2.7), we obtain

$$\begin{aligned}
& \langle (P\Delta_0(\vartheta)P + P_1\Delta_1P_1)f, f \rangle_{V^c} = \\
&= \sum_{v \in V_0^c \setminus V_{01}} \sum_{\mathbf{e}=(v,u) \in \mathcal{A}_0^c} (f(v) - e^{i\langle \tau(\mathbf{e}), \vartheta \rangle} f(u)) \bar{f}(v) + \sum_{v \in V_1 \setminus V_{01}} \sum_{(v,u) \in \mathcal{A}_1} (f(v) - f(u)) \bar{f}(v) \\
&+ \sum_{v \in V_{01}} \left(\sum_{\mathbf{e}=(v,u) \in \mathcal{A}_0^c} (f(v) - e^{i\langle \tau(\mathbf{e}), \vartheta \rangle} f(u)) + \sum_{(v,u) \in \mathcal{A}_1} (f(v) - f(u)) \right) \bar{f}(v) \\
&= \sum_{v \in V^c} \sum_{\mathbf{e}=(v,u) \in \mathcal{A}^c} (f(v) - e^{i\langle \tau(\mathbf{e}), \vartheta \rangle} f(u)) \bar{f}(v) = \sum_{v \in V^c} (\Delta(\vartheta)f)(v) \bar{f}(v) = \langle \Delta(\vartheta)f, f \rangle_{V^c},
\end{aligned}$$

which implies (3.7). \blacksquare

Proof of Proposition 2.1.ii) For each $\vartheta \in \mathbb{T}^d$ the unperturbed fiber Laplacian $\Delta_0(\vartheta)$ is $\mathbb{Z}^{\tilde{d}-d}$ -periodic. Then, using standard arguments (see Theorem XIII.85 in [RS78]), we obtain that the spectrum of $\Delta_0(\vartheta)$ has the form

$$\sigma(\Delta_0(\vartheta)) = \sigma_{ac}(\Delta_0(\vartheta)) \cup \sigma_{fb}(\Delta_0(\vartheta)).$$

Since the graph Γ_1 is finite, for each $\lambda \in \sigma_{fb}(\Delta_0(\vartheta))$ there exists a corresponding eigenfunction $f \in \ell^2(V_0^c)$ with a finite support (see, e.g., Theorem 4.5.2 in [BK13]) not intersecting with V_1 . Due to (3.7), $(f, 0) \in \ell^2(V^c)$ is an eigenfunction of $\Delta(\vartheta)$ with the same finite support and the same eigenvalue λ . Thus, $\lambda \in \sigma_{fb}(\Delta(\vartheta))$ and vice versa. Since the operator Δ_1 has finite rank p , where p is defined in (2.10), for each $\vartheta \in \mathbb{T}^d$ the spectrum $\sigma(\Delta(\vartheta))$ of the fiber Laplacian $\Delta(\vartheta)$ is given by (2.8), where $\sigma_{ac}(\Delta(\vartheta))$, $\sigma_{fb}(\Delta(\vartheta))$ satisfy (2.9) and $\sigma_p(\Delta(\vartheta))$ consists of $N_\vartheta \leq p$ eigenvalues (2.10). \blacksquare

4. PROOF OF THE MAIN RESULTS

In this section we prove Theorem 2.3 about the position of guided bands and Theorem 2.4 about the asymptotics of the guided bands for guides with large multiplicity of their edges. We prove Corollary 2.5 about geometric properties of the guided spectrum for periodic graphs with specific guides.

4.1. Estimates for the guided spectrum. Denote by $m_\pm(\vartheta)$ the upper and lower endpoints of the spectrum of the unperturbed fiber Laplacian $\Delta_0(\vartheta)$:

$$m_-(\vartheta) = \inf \sigma(\Delta_0(\vartheta)), \quad m_+(\vartheta) = \sup \sigma(\Delta_0(\vartheta)). \quad (4.1)$$

Then (1.5) yields

$$\min_{\vartheta \in \mathbb{T}^d} m_-(\vartheta) = 0, \quad \max_{\vartheta \in \mathbb{T}^d} m_+(\vartheta) = \varrho. \quad (4.2)$$

We need a simple estimate for eigenvalues of bounded self-adjoint operators [RS78]: *Let A, B be bounded self-adjoint operators in a Hilbert space \mathcal{H} and let $\lambda_j(A) = \max \{\tilde{\lambda}_j(A), \sup \sigma_{ess}(A)\}$, $j = 1, 2, \dots$, where $\tilde{\lambda}_1(A) \geq \tilde{\lambda}_2(A) \geq \dots$ are the eigenvalues of A . Then*

$$\lambda_j(A) + \inf \sigma(B) \leq \lambda_j(A + B) \leq \lambda_j(A) + \sup \sigma(B), \quad j = 1, 2, 3, \dots \quad (4.3)$$

The following simple corollary about the position of the guided bands $\mathfrak{s}_j^o(\Delta)$ defined by (2.14) is a direct consequence of Proposition 2.1.

Corollary 4.1. *Let Δ be the Laplacian on a perturbed graph Γ and let $\varrho = \sup \sigma(\Delta_0)$. Then each guided band $\mathfrak{s}_j^o(\Delta)$, $j = 1, \dots, N_g$, and their number N_g satisfy*

$$\mathfrak{s}_j^o(\Delta) \subset [\zeta_j, \zeta_j + \varrho], \quad (4.4)$$

$$N_g \geq \#\{j \in \mathbb{N}_p : \zeta_j > \varrho\}, \quad (4.5)$$

where $\zeta_1 \geq \dots \geq \zeta_p$ are the positive eigenvalues of the Laplacian Δ_1 and $p = \text{rank } \Delta_1$.

Proof. Each fiber Laplacian $\Delta(\vartheta)$, $\vartheta \in \mathbb{T}^d$, is given by (3.7). Then, due to (4.1) – (4.3), each eigenvalue $\lambda_j(\vartheta)$ of $\Delta(\vartheta)$ above its essential spectrum satisfy

$$\zeta_j \leq \zeta_j + m_-(\vartheta) \leq \lambda_j(\vartheta) \leq \zeta_j + m_+(\vartheta) \leq \zeta_j + \varrho, \quad (4.6)$$

which yields (4.4). Let $\zeta_j > \varrho$ for some $j = 1, \dots, p$. Then, due to (4.6), $\lambda_j(\vartheta) > \varrho$ for all $\vartheta \in \mathbb{T}^d$. Thus, λ_j creates the guided band $\mathfrak{s}_j^o(\Delta) = \lambda_j(\mathbb{T}^d)$. This yields (4.5). ■

Remark. It is well known, see, e.g., [F73], that the positive eigenvalues $\zeta_1 \geq \dots \geq \zeta_p$ of the Laplacian Δ_1 on a finite graph $\Gamma_1 = (V_1, \mathcal{E}_1)$ satisfy

$$\frac{\nu_1}{\nu_1 - 1} \max_{v \in V_1} \kappa_v^1 \leq \zeta_1 \leq \max_{\substack{u, v \in V_1 \\ u \sim v}} (\kappa_u^1 + \kappa_v^1),$$

$$\min_{\substack{u, v \in V_1 \\ u \sim v}} (\kappa_u^1 + \kappa_v^1) - (\nu_1 - 2) \leq \zeta_p \leq \frac{\nu_1}{\nu_1 - 1} \min_{v \in V_1} \kappa_v^1,$$

where $\nu_1 = \#V_1$, κ_v^1 is the degree of the vertex $v \in V_1$ on Γ_1 . From these estimates and (4.4) it follows that increasing the degree of at least one vertex of the graph Γ_1 removes the first guided band $\mathfrak{s}_1^o(\Delta)$ arbitrarily far to the right. Increasing the degrees of all vertices of Γ_1 removes all guided bands arbitrarily far to the right.

Proof of Theorem 2.3. i) We rewrite the fiber Laplacian $\Delta(\vartheta)$, $\vartheta \in \mathbb{T}^d$, defined by (2.7) in the form:

$$\Delta(\vartheta) = \Delta^m + \Delta_\beta(\vartheta), \quad (4.7)$$

$$(\Delta_\beta(\vartheta)f)(v) = \sum_{\substack{\mathbf{e}=(v,u) \in \mathcal{A}_c \\ \tau(\mathbf{e}) \neq 0}} (f(v) - e^{i\langle \tau(\mathbf{e}), \vartheta \rangle} f(u)), \quad v \in V^c, \quad (4.8)$$

where $\tau(\mathbf{e}) \in \mathbb{Z}^d$ is the index of the edge $\mathbf{e} \in \mathcal{A}^c$ defined by (3.3), (3.4). Each operator $\Delta_\beta(\vartheta)$, $\vartheta \in \mathbb{T}^d$, is the magnetic Laplacian on the graph $\mathcal{C}_\beta = (V^c, \mathcal{B}^c)$ and the degree of each vertex $v \in V^c$ on \mathcal{C}_β is equal to the number β_v of all bridges starting at v . Then the spectrum $\sigma(\Delta_\beta(\vartheta)) \subset [0, 2\beta_+]$ for each $\vartheta \in \mathbb{T}^d$ (see, e.g., [HS99]) and, due to (4.3), each eigenvalue $\lambda_j(\vartheta)$ of $\Delta(\vartheta)$ above its essential spectrum satisfy $\mu_j \leq \lambda_j(\vartheta) \leq \mu_j + 2\beta_+$, which yields (2.18).

ii) The existence of such graphs is proved in Propositions 5.2 and 5.3. ■

4.2. Proof of Theorem 2.4. Let $\Gamma_t = (V_1, \mathcal{E}_t)$ be a finite graph obtained from the graph $\Gamma_1 = (V_1, \mathcal{E}_1)$ considering each edge of Γ_1 to have the multiplicity $t \in \mathbb{N}$. Then the Laplacian on the graph Γ_t has the form $t\Delta_1$. If t is large enough, then all positive eigenvalues $t\zeta_p \leq \dots \leq t\zeta_2 \leq t\zeta_1$ of the Laplacian $t\Delta_1$ on the graph Γ_t satisfy $t\zeta_p > \varrho$ and $t(\zeta_j - \zeta_{j+1}) > \varrho$ for all $j \in \mathbb{N}_{p-1}$, where ϱ is defined in (1.5). Then, due to Corollary 4.1, the guided spectrum of the Laplacian Δ_t consists of exactly p guided bands separated by gaps.

i) If $f_j = 0$ on V_{01} , then, due to Proposition 2.2, $\{t\zeta_j\}$ is a guided flat band of the Laplacian Δ_t on the perturbed graph $\Gamma = \Gamma_0 \cup \Gamma_t^g$.

Let $f_j = (f_{j,01}, f_{j,1}) \in \ell^2(V_1)$, where $0 \neq f_{j,01} \in \ell^2(V_{01})$ and $f_{j,1} \in \ell^2(V_1 \setminus V_{01})$. Using (3.7), we rewrite the fiber Laplacian $\Delta_t(\vartheta)$, $\vartheta \in \mathbb{T}^d$, for the Laplacian Δ_t acting on the perturbed graph $\Gamma = \Gamma_0 \cup \Gamma_t^g$ in the form

$$\Delta_t(\vartheta) = P\Delta_0(\vartheta)P + tP_1\Delta_1P_1 = tK_t(\vartheta), \quad K_t(\vartheta) = P_1\Delta_1P_1 + \varepsilon P\Delta_0(\vartheta)P, \quad \varepsilon = \frac{1}{t}.$$

We denote the eigenvalues of the operator $K_t(\vartheta)$ above its essential spectrum by

$$E_p(\vartheta, t) \leq \dots \leq E_2(\vartheta, t) \leq E_1(\vartheta, t), \quad \vartheta \in \mathbb{T}^d. \quad (4.9)$$

Then the eigenvalue $E_j(\vartheta, t)$ of the operator $K_t(\vartheta)$ has the following asymptotics:

$$E_j(\vartheta, t) = \zeta_j + \varepsilon W_j(\vartheta) + O(\varepsilon^2) \quad (4.10)$$

(see pp. 7–8 in [RS78]) uniformly in $\vartheta \in \mathbb{T}^d$ as $t \rightarrow \infty$, where

$$W_j(\vartheta) = \langle (0, f_{j,01}), \Delta_0(\vartheta)(0, f_{j,01}) \rangle_{V_0^c}, \quad (4.11)$$

$\langle \cdot, \cdot \rangle_{V_0^c}$ denotes the inner product in $\ell^2(V_0^c)$. This yields the asymptotics of the eigenvalue $\lambda_j(\vartheta, t)$ of the operator $\Delta_t(\vartheta)$:

$$\lambda_j(\vartheta, t) = t E_j(\vartheta, t) = t\zeta_j + W_j(\vartheta) + O(1/t). \quad (4.12)$$

Using this asymptotics for $\lambda_j^-(t) = \min_{\vartheta \in \mathbb{T}^d} \lambda_j(\vartheta, t)$ and $\lambda_j^+(t) = \max_{\vartheta \in \mathbb{T}^d} \lambda_j(\vartheta, t)$, we obtain

$$\lambda_j^\pm(t) = t\zeta_j + W_j^\pm + O(1/t),$$

where W_j^\pm are defined in (2.20). Since $\mathfrak{s}_j(\Delta_t) = [\lambda_j^-(t), \lambda_j^+(t)]$, the asymptotics (4.12) also gives the second formula in (2.19). Using the formula (3.9) for the fiber Laplacian $\Delta_0(\vartheta)$, we obtain

$$\begin{aligned} W_j(\vartheta) &= \sum_{v \in V_{01}} (\Delta_0(\vartheta)(0, f_{j,01}))(v) \bar{f}_j(v) = \sum_{v \in V_{01}} \sum_{\mathbf{e}=(v,u) \in \mathcal{A}_0^c} (f_j(v) - e^{i\langle \tau(\mathbf{e}), \vartheta \rangle} f_j(u)) \bar{f}_j(v) \\ &= \sum_{v \in V_{01}} \kappa_v^0 |f_j(v)|^2 - \sum_{\substack{\mathbf{e}=(v,u) \in \mathcal{A}_0^c \\ v,u \in V_{01}}} e^{i\langle \tau(\mathbf{e}), \vartheta \rangle} f_j(u) \bar{f}_j(v) \\ &= \sum_{v \in V_{01}} \kappa_v^0 |f_j(v)|^2 - \sum_{\substack{\mathbf{e}=(v,u) \in \mathcal{A}_0^c \\ v,u \in V_{01}}} \cos \langle \tau(\mathbf{e}), \vartheta \rangle f_j(u) \bar{f}_j(v), \end{aligned} \quad (4.13)$$

where κ_v^0 is the degree of the vertex v on the unperturbed cylinder \mathcal{C}_0 and $\tau(\mathbf{e}) \in \mathbb{Z}^d$ is the edge index defined by (3.3), (3.4). Using this formula we rewrite the constant W_j^\bullet defined in

(2.20) in the form

$$W_j^\bullet = \max_{\vartheta \in \mathbb{T}^d} \Omega_j(\vartheta) - \min_{\vartheta \in \mathbb{T}^d} \Omega_j(\vartheta), \quad \Omega_j(\vartheta) = \sum_{\substack{\mathbf{e}=(v,u) \in \mathcal{A}_0^c \\ v,u \in V_{01}, \tau(\mathbf{e}) \neq 0}} \cos \langle \tau(\mathbf{e}), \vartheta \rangle f_j(u) f_j(v). \quad (4.14)$$

We have

$$|\Omega_j(\vartheta)| \leq \sum_{\substack{\mathbf{e}=(v,u) \in \mathcal{A}_0^c \\ v,u \in V_{01}, \tau(\mathbf{e}) \neq 0}} |\cos \langle \tau(\mathbf{e}), \vartheta \rangle| \leq \beta_{01},$$

which yields $W_j^\bullet \leq 2\beta_{01}$.

ii) Let $V_{01} = \{v\}$. Then for Ω_j defined in (4.14) we have

$$\Omega_j(\vartheta) = f_j^2(v) \sum_{\substack{\mathbf{e}=(v,v) \in \mathcal{A}_0^c \\ \tau(\mathbf{e}) \neq 0}} \cos \langle \tau(\mathbf{e}), \vartheta \rangle, \quad \max_{\vartheta \in \mathbb{T}^d} \Omega_j(\vartheta) = f_j^2(v) \beta_{01}. \quad (4.15)$$

Using the identity $\int_{\mathbb{T}^d} \cos \langle \tau(\mathbf{e}), \vartheta \rangle d\vartheta = 0$ for each $\tau(\mathbf{e}) \neq 0$, we obtain

$$-f_j^2(v) \beta_{01} \leq \min_{\vartheta \in \mathbb{T}^d} \Omega_j(\vartheta) \leq 0. \quad (4.16)$$

Then (4.14) – (4.16) yield $f_j^2(v) \beta_{01} \leq W_j^\bullet \leq 2f_j^2(v) \beta_{01}$. From (4.15) and the condition $f_j(v) \neq 0$ it follows that $\Omega_j(\cdot) = \text{const}$ iff $\beta_{01} = 0$. This yields the last statement of the item.

iii) If all positive eigenvalues $\zeta_1 > \dots > \zeta_p$ of the Laplacian Δ_1 are distinct, then summing the second asymptotics in (2.19) over $j = 1, \dots, p$ we obtain (2.22).

iv) If there is no bridge connecting the vertices from the set V_{01} on the cylinder \mathcal{C} , then for each $j \in \mathbb{N}_p$ the function W_j defined by (4.13) is constant, i.e., $W_j^\bullet = 0$, and, the second asymptotics in (2.19) and the asymptotics (2.22), take the form $|\mathfrak{s}_j(\Delta_t)| = O(1/t)$ and $|\mathfrak{s}(\Delta_t)| = O(1/t)$, respectively. ■

Remark. The set \mathcal{A}_0^c of all oriented edges of the cylinder \mathcal{C}_0 is infinite, but the sum in (4.13) is taken over a finite (maybe empty) set of edges $(v, u) \in \mathcal{A}_0^c$ for which $v, u \in V_{01}$.

4.3. Geometric properties of the guided spectrum. Now we prove Proposition 2.2 and Corollary 2.5 about geometric properties of the guided spectrum for specific graphs.

Proof of Proposition 2.2. Let ζ be an eigenvalue of the Laplacian Δ_1 on Γ_1 with an eigenfunction $f \in \ell^2(V_1)$ equal to zero on V_{01} . Then, due to (3.7), for each $\vartheta \in \mathbb{T}^d$ the function $(0, f) \in \ell^2(V^c)$ is an eigenfunction of the fiber Laplacian $\Delta(\vartheta)$ associated with the same eigenvalue ζ . Thus, $\{\zeta\}$ is a guided flat band of the Laplacian Δ . ■

Remarks. 1) The $\nu_1 \times \nu_1$ matrix $\Delta_1 = \{\Delta_{uv}^1\}_{u,v \in V_1}$ associated to the Laplacian Δ_1 on a finite graph $\Gamma_1 = (V_1, \mathcal{E}_1)$ in the standard orthonormal basis is given by

$$\Delta_{uv}^1 = \delta_{uv} \nu_v^1 - \nu_{uv}, \quad \nu_1 = \#V_1, \quad (4.17)$$

where δ_{uv} is the Kronecker delta, ν_v^1 is the degree of the vertex $v \in V_1$ on the graph Γ_1 , ν_{uv} is the number of edges $(u, v) \in \mathcal{A}_1$.

2) The sufficient conditions from Proposition 2.2 are equivalent to ζ being an eigenvalue of two operators: the Laplacian Δ_1 and the Laplacian Δ_D on Γ_1 with Dirichlet boundary condition

$$f(v) = 0, \quad \forall v \in V_{01}, \quad (4.18)$$

i.e., ζ is an eigenvalue of two matrices: $\Delta_1 = \{\Delta_{uv}^1\}_{u,v \in V_1}$ and its submatrix $\Delta_D = \{\Delta_{uv}^1\}_{u,v \in V_1 \setminus V_{01}}$.

Proof of Corollary 2.5. i) Due to the connectivity of the periodic graph Γ_0 , there exists a bridge on the cylinder \mathcal{C}_0 . First we consider the case when there exists a bridge-loop at some vertex $u_1 \in V_0^c$. Due to the periodicity of the cylinder \mathcal{C}_0 , for each $j \in \mathbb{Z}$ we have $u_j = u_1 + ja_{\tilde{d}} \in V_0^c$, where $a_{\tilde{d}}$ is one of the periods of \mathcal{C}_0 .

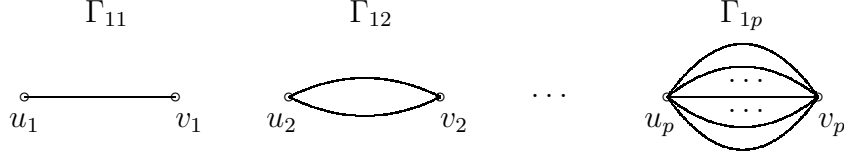


FIGURE 3. A finite graph Γ_1 with p connected components $\Gamma_{11}, \dots, \Gamma_{1p}$.

Let $\Gamma_1 = (V_1, \mathcal{E}_1)$ be a finite graph consisting of $p \in \mathbb{N}$ connected components $\Gamma_{11}, \dots, \Gamma_{1p}$, where Γ_{1j} is a graph consisting of two vertices $u_j \in V_0^c$, $v_j \notin V_0^c$ and j edges connecting these vertices (see Fig.3). The Laplacian Δ_1 on the finite graph Γ_1 has exactly p simple positive eigenvalues

$$\zeta_1 = 2p, \quad \zeta_2 = 2(p-1), \quad \dots, \quad \zeta_p = 2.$$

The normalized eigenfunction $f_j \in \ell^2(V_1)$ of the Laplacian Δ_1 corresponding to the eigenvalue ζ_j has the form

$$f_j(v) = \begin{cases} \frac{1}{\sqrt{2}}, & \text{if } v = u_j \\ -\frac{1}{\sqrt{2}}, & \text{if } v = v_j \\ 0, & \text{otherwise} \end{cases}, \quad j = 1, \dots, p. \quad (4.19)$$

Let Γ_t , $t \in \mathbb{N}$, be a finite graph obtained from the graph Γ_1 considering each edge of Γ_1 to have the multiplicity t . Let Δ_t be the Laplacian on the perturbed graph $\Gamma = \Gamma_0 \cup \Gamma_t^g$. Due to Theorem 2.4, for t large enough the guided spectrum of the Laplacian Δ_t consists of exactly p guided bands $\mathfrak{s}_j^o(\Delta_t) = \mathfrak{s}_j(\Delta_t)$ separated by gaps and these bands satisfy

$$|\mathfrak{s}_j(\Delta_t)| = W_j^\bullet + O(1/t), \quad j \in \mathbb{N}_p, \quad (4.20)$$

where W_j^\bullet is defined in (2.20). Substituting the identities (4.19) into (4.13), we obtain the following expression for the function W_j :

$$W_j(\vartheta) = \frac{\kappa_{u_j}^0}{2} - \sum_{\mathbf{e}=(u_j, u_j) \in \mathcal{A}_0^c} \cos\langle \tau(\mathbf{e}), \vartheta \rangle. \quad (4.21)$$

Due to the periodicity of the cylinder \mathcal{C}_0 , for each $\vartheta \in \mathbb{T}^d$ we have

$$\sum_{\mathbf{e}=(u_1, u_1) \in \mathcal{A}_0^c} \cos\langle \tau(\mathbf{e}), \vartheta \rangle = \sum_{\mathbf{e}=(u_2, u_2) \in \mathcal{A}_0^c} \cos\langle \tau(\mathbf{e}), \vartheta \rangle = \dots \quad (4.22)$$

This and (4.21) yield that $W_1^\bullet = \dots = W_p^\bullet \neq 0$. Then, due to (4.20), all guided bands are non-degenerate and

$$|\mathfrak{s}(\Delta_t)| = \sum_{j=1}^p |\mathfrak{s}_j(\Delta_t)| = p W_1^\bullet + O(1/t). \quad (4.23)$$

Choosing $p > \frac{C}{W_1^\bullet}$, we obtain $|\mathfrak{s}(\Delta_t)| > C$ for t large enough.

Now let there be no bridge-loop on the cylinder \mathcal{C}_0 . Then the existing bridge connects some distinct vertices $u_1, v_1 \in V_0^c$. Repeating the above arguments but with $u_j = u_1 + ja_{\tilde{d}} \in V_0^c$ and $v_j = v_1 + ja_{\tilde{d}} \in V_0^c$, $j \in \mathbb{N}_p$, we also obtain the required statement. Note that in this case the function W_j has the form

$$W_j(\vartheta) = \frac{1}{2}(\kappa_{u_j}^0 + \kappa_{v_j}^0) + \sum_{\mathbf{e}=(u_j, v_j) \in \mathcal{A}_0^c} \cos\langle \tau(\mathbf{e}), \vartheta \rangle, \quad j \in \mathbb{N}_p. \quad (4.24)$$

ii) Let there exist a vertex $v \in V_0^c$ such that there is no bridge on \mathcal{C}_0 starting at v . We consider a finite graph $\Gamma_t = (V_1, \mathcal{E}_t)$ consisting of two vertices $u \notin V_0^c$ and v and the edge (u, v) of multiplicity $t \in \mathbb{N}$. The Laplacian on the finite graph Γ_t has one simple positive eigenvalue $\zeta_1 = 2t$. Let Δ_t be the Laplacian on the perturbed graph $\Gamma = \Gamma_0 \cup \Gamma_t^g$. Due to Theorem 2.4, for t large enough the guided spectrum of Δ_t consists of exactly one guided band $\mathfrak{s}_1(\Delta_t)$ and the length of this guided band satisfies $|\mathfrak{s}_1(\Delta_t)| = O(1/t)$. Thus, for any small $\varepsilon > 0$ there exists $t \in \mathbb{N}$ such that $|\mathfrak{s}(\Delta_t)| < \varepsilon$.

iii) This follows from (2.16) and the fact that the eigenvalues of the Laplacian on the finite graph Γ_t can be arbitrary large as $t \rightarrow \infty$.

iv) The sufficient conditions for the existence of guided flat bands are proved in Proposition 2.2. For examples of finite graphs Γ_1 satisfying these conditions see Propositions 5.2, 5.3. ■

4.4. Reduction to operators on unperturbed cylinder. We reduce the eigenvalue problem for the fiber Laplacian on the perturbed cylinder \mathcal{C} to that on the unperturbed cylinder \mathcal{C}_0 . In order to do this we use the following well-known theorem [BFS98].

Theorem 4.2 (Feshbach projection method). *Let P be an orthogonal projection on a separable Hilbert space \mathcal{H} , and let $P^\perp = \mathbb{1} - P$ be its complement. Let T be a bounded self-adjoint operator. Assume that $P^\perp T P^\perp$ is invertible on $P^\perp \mathcal{H}$. Then*

i) *T is invertible on \mathcal{H} if and only if its Feshbach map*

$$\mathcal{F} = PTP - PTP^\perp(P^\perp TP^\perp)^{-1}P^\perp TP \quad (4.25)$$

is invertible on $P\mathcal{H}$; in this case $\mathcal{F}^{-1} = PT^{-1}P$;

ii) *if $T\psi = 0$ for some vector $0 \neq \psi \in \mathcal{H}$, then $\mathcal{F}P\psi = 0$, where $P\psi \neq 0$;*

iii) *if $\mathcal{F}\varphi = 0$ for some vector $0 \neq \varphi = P\varphi$, then $T\psi = 0$, where*

$$0 \neq \psi = [P - P^\perp(P^\perp TP^\perp)^{-1}P^\perp TP]\varphi;$$

iv) *the kernels of T and \mathcal{F} have equal dimensions.*

Remark. The operator T in our consideration is $\Delta(\vartheta) - \lambda$, where $\Delta(\vartheta)$ is the fiber Laplacian on the perturbed cylinder $\mathcal{C} = (V^c, \mathcal{E}^c)$.

Proposition 4.3. *Let P be the orthogonal projection of $\ell^2(V^c)$ onto the subspace $\ell^2(V_0^c)$. Then the following statements hold true.*

i) *If $P^\perp(\Delta_1 - \lambda)P^\perp$ is invertible on $P^\perp \ell^2(V^c)$, then for each $\vartheta \in \mathbb{T}^d$*

$$\lambda \in \sigma_p(\Delta(\vartheta)) \quad \Leftrightarrow \quad 0 \in \sigma_p(\mathcal{F}(\vartheta, \lambda)), \quad (4.26)$$

and the kernels of $\Delta(\vartheta) - \lambda$ and $\mathcal{F}(\vartheta, \lambda)$ have equal dimensions. Here $\mathcal{F}(\vartheta, \lambda)$ is the Feshbach map (4.25) for the operator $\Delta(\vartheta) - \lambda$, defined by (2.7), and $\mathcal{F}(\vartheta, \lambda)$ has the form

$$\mathcal{F}(\vartheta, \lambda) = P(\Delta_0(\vartheta) - \lambda)P + P_{01}(\Delta_1 - \Delta_1 P^\perp(P^\perp(\Delta_1 - \lambda)P^\perp)^{-1}P^\perp \Delta_1)P_{01}, \quad (4.27)$$

P_{01} is the orthogonal projection of $\ell^2(V^c)$ onto $\ell^2(V_{01})$.

ii) Let, in addition, a finite graph Γ_1 consist of $c := c_{\Gamma_1}$ connected components

$$\Gamma_{11} = (V_{11}, \mathcal{E}_{11}), \quad \dots, \quad \Gamma_{1c} = (V_{1c}, \mathcal{E}_{1c})$$

each of which has exactly one common vertex v_1, \dots, v_c , respectively, with the unperturbed cylinder $\mathcal{C}_0 = (V_0^c, \mathcal{E}_0^c)$. Then for each $\vartheta \in \mathbb{T}^d$ the Feshbach map (4.27) for the operator $\Delta(\vartheta) - \lambda$ is just a fiber Schrödinger operator given by

$$\mathcal{F}(\vartheta, \lambda) = P(\Delta_0(\vartheta) - \lambda)P + Q(\lambda), \quad (4.28)$$

where $Q(\lambda) = Q(\lambda, \cdot)$ is a potential with the compact support $V_{01} = \{v_1, \dots, v_c\}$:

$$Q(\lambda, v_j) = (\mathcal{P}_j(\Delta_{1j} - \Delta_{1j}\mathcal{P}_j^\perp(\mathcal{P}_j^\perp(\Delta_{1j} - \lambda)\mathcal{P}_j^\perp)^{-1}\mathcal{P}_j^\perp\Delta_{1j})\mathcal{P}_j) \upharpoonright_{\ell^2(\{v_j\})}, \quad j = 1, \dots, c, \quad (4.29)$$

Δ_{1j} is the Laplacian on the finite graph Γ_{1j} , \mathcal{P}_j is the orthogonal projection of $\ell^2(V_{1j})$ onto the one-dimensional subspace $\ell^2(\{v_j\})$. In particular, if the operator $\Delta_{1j} - \lambda\mathcal{P}_j^\perp$ is invertible, then the expression (4.29) can be written in the form

$$Q^{-1}(\lambda, v_j) = (\mathcal{P}_j(\Delta_{1j} - \lambda\mathcal{P}_j^\perp)^{-1}\mathcal{P}_j) \upharpoonright_{\ell^2(\{v_j\})}. \quad (4.30)$$

iii) If $P^\perp(\Delta_1 - \lambda)P^\perp$ is not invertible on $P^\perp\ell^2(V^c)$, then λ is a guided flat band of the Laplacian Δ on the perturbed graph $\Gamma = \Gamma_0 \cup \Gamma_1^g$.

Proof. i) Let P_1 be the orthogonal projection of $\ell^2(V^c)$ onto $\ell^2(V_1)$. Then, using (4.25), (3.7) and the identities

$$PP_1 = P_1P = P_{01}, \quad PP^\perp = P^\perp P = 0, \quad P^\perp P_1 = P_1P^\perp = P^\perp, \quad (4.31)$$

we have

$$\begin{aligned} \mathcal{F}(\vartheta, \lambda) &= P(\Delta(\vartheta) - \lambda)P - P\Delta(\vartheta)P^\perp(P^\perp(\Delta(\vartheta) - \lambda)P^\perp)^{-1}P^\perp\Delta(\vartheta)P \\ &= P(P\Delta_0(\vartheta)P + P_1\Delta_1P_1 - \lambda)P \\ &\quad - P(P\Delta_0(\vartheta)P + P_1\Delta_1P_1)P^\perp(P^\perp(P\Delta_0(\vartheta)P + P_1\Delta_1P_1 - \lambda)P^\perp)^{-1}P^\perp(P\Delta_0(\vartheta)P + P_1\Delta_1P_1)P \\ &= P(\Delta_0(\vartheta) - \lambda)P + P_{01}\Delta_1P_{01} - P_{01}\Delta_1P^\perp(P^\perp(\Delta_1 - \lambda)P^\perp)^{-1}P^\perp\Delta_1P_{01}. \end{aligned} \quad (4.32)$$

This and Theorem 4.2 yield the required statement.

ii) In this case $\Delta_1 = \Delta_{11} \oplus \dots \oplus \Delta_{1c}$. Then (4.27) has the form (4.28), (4.29). Let the operator $\Delta_{1j} - \lambda\mathcal{P}_j^\perp$ be invertible. Then, using the partitioned presentation of the inverse matrix (see p.18 in [HJ85]), we obtain (4.30).

iii) Let $P^\perp(\Delta_1 - \lambda)P^\perp$ not be invertible on $P^\perp\ell^2(V^c)$, then λ is an eigenvalue of the operator $P^\perp\Delta_1P^\perp$ and, due to Proposition 2.2, $\{\lambda\}$ is a guided flat band of the Laplacian Δ on the perturbed graph Γ . ■

5. SQUARE LATTICE WITH GUIDES.

In this section we consider the square lattice with specific guides. We obtain some properties of the guided spectrum on such graphs and give examples of the guided spectrum.

Let $\mathbb{L}^2 = (V, \mathcal{E})$ be the square lattice, where the vertex set $V = \mathbb{Z}^2$ and the edge set $\mathcal{E} = \{(m, m + (1, 0)), (m, m + (0, 1)), \forall m \in \mathbb{Z}^2\}$, see Fig.4.a. The Laplacian Δ_0 on \mathbb{L}^2 has

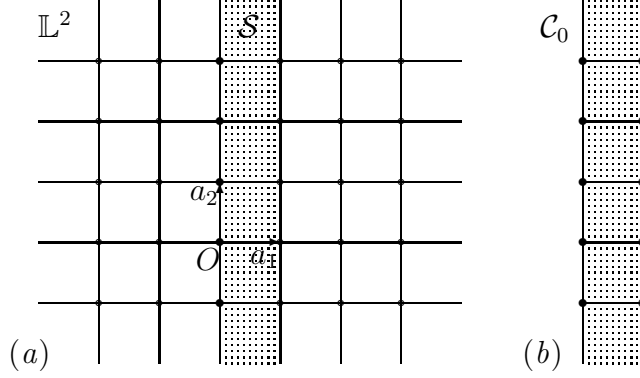


FIGURE 4. *a)* The square lattice \mathbb{L}^2 ; the vertices from the fundamental vertex set V_0^c are big black points; the strip \mathcal{S} is shaded; *b)* the cylinder $\mathcal{C}_0 = \mathbb{L}^2/\mathbb{Z}$ (the edges of the strip are identified).

the form

$$(\Delta_0 f)(m) = 4f(m) - \sum_{|m-k|=1} f(k), \quad f \in \ell^2(\mathbb{Z}^2), \quad m \in \mathbb{Z}^2. \quad (5.1)$$

We consider the Laplacian Δ on the perturbed square lattice $\Gamma = \mathbb{L}^2 \cup \Gamma_1^g$ with a guide Γ_1^g . Due to Proposition 2.1, the Laplacian Δ on Γ has the decomposition (2.6), (3.7) into a constant fiber direct integral, where the fiber unperturbed Laplacian $\Delta_0(\vartheta)$ acts on $f \in \ell^2(\mathbb{Z})$ and is given by

$$\begin{aligned} \Delta_0(\vartheta) &= 2(1 - \cos \vartheta) + h, \\ (hf)(n) &= 2f(n) - f(n+1) - f(n-1), \quad n \in \mathbb{Z}, \end{aligned} \quad (5.2)$$

for all $\vartheta \in \mathbb{T} = (-\pi, \pi]$. It is well known that the spectrum of the Laplacian h on \mathbb{Z} is given by $\sigma(h) = \sigma_{ac}(h) = [0, 4]$. Then the spectrum of each fiber Laplacian $\Delta(\vartheta)$, $\vartheta \in \mathbb{T}$, has the form

$$\begin{aligned} \sigma(\Delta(\vartheta)) &= \sigma_{ac}(\Delta(\vartheta)) \cup \sigma_p(\Delta(\vartheta)), \\ \sigma_{ac}(\Delta(\vartheta)) &= \sigma_{ac}(\Delta_0(\vartheta)) = [2 - 2\cos \vartheta, 6 - 2\cos \vartheta], \end{aligned} \quad (5.3)$$

$\sigma_p(\Delta(\vartheta))$ is the set of all eigenvalues of $\Delta(\vartheta)$ of finite multiplicity given by (2.10).

The spectrum of the Laplacian Δ on the perturbed square lattice $\Gamma = \mathbb{L}^2 \cup \Gamma_1^g$ has the form

$$\sigma(\Delta) = \sigma(\Delta_0) \cup \mathfrak{s}(\Delta), \quad \sigma(\Delta_0) = [0, 8], \quad \mathfrak{s}(\Delta) = \bigcup_{j=1}^N \mathfrak{s}_j(\Delta) = \mathfrak{s}_{ac}(\Delta) \cup \mathfrak{s}_{fb}(\Delta),$$

where N is defined in (2.11).

Now we consider the perturbed square lattice $\Gamma = \mathbb{L}^2 \cup \Gamma_1^g$ when a finite graph Γ_1 has exactly one common vertex $v_1 = 0$ with the unperturbed cylinder $\mathcal{C}_0 = \mathbb{L}^2/\mathbb{Z} = (\mathbb{Z}, \mathcal{E}_0^c)$, see Fig. 5.

Proposition 5.1. *Let $\Gamma_1 = (V_1, \mathcal{E}_1)$ be a finite connected graph and let $V_{01} = \mathbb{Z} \cap V_1 = \{0\}$. Then the guided spectrum $\mathfrak{s}_{ac}(\Delta)$ and $\mathfrak{s}_{fb}(\Delta)$ of the Laplacian Δ on the perturbed square lattice $\Gamma = \mathbb{L}^2 \cup \Gamma_1^g$ satisfy*

$$\mathfrak{s}_{ac}(\Delta) = \left\{ \lambda \in \mathbb{R} : 2 \leq \lambda - \sqrt{4 + Q^2(\lambda)} \leq 6, \quad Q(\lambda) > 0 \right\}, \quad (5.4)$$

$$\mathfrak{s}_{fb}(\Delta) = \left\{ \lambda \in \mathbb{R} : \lambda \text{ is an eigenvalue of the operator } \mathcal{P}\Delta_1\mathcal{P} \right\}, \quad (5.5)$$

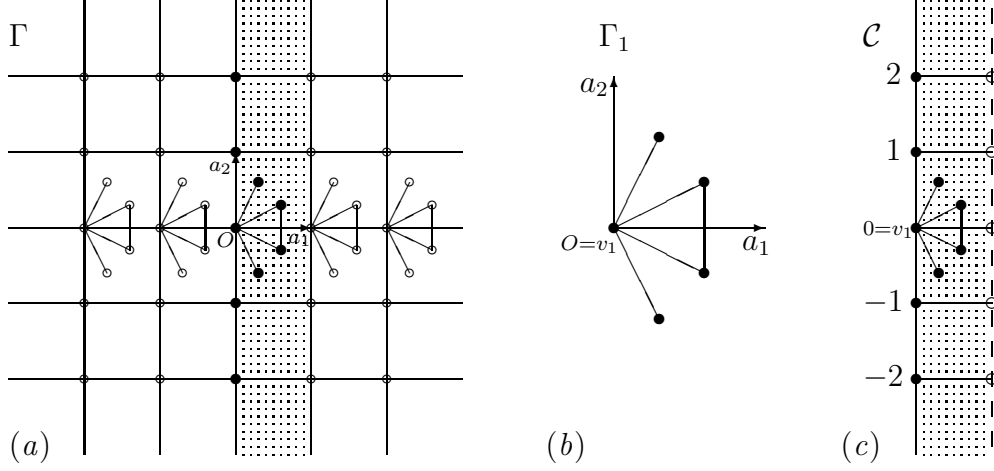


FIGURE 5. *a)* The perturbed square lattice $\Gamma = \mathbb{L}^2 \cup \Gamma_1^g$; *b)* a finite graph Γ_1 ; *c)* the perturbed cylinder $\mathcal{C} = \Gamma/\mathbb{Z}$.

where \mathcal{P} is the orthogonal projection of $\ell^2(V_1)$ onto the one-dimensional subspace $\ell^2(\{0\})$, Δ_1 is the Laplacian on the finite graph Γ_1 ,

$$Q(\lambda) = (\mathcal{P}(\Delta_1 - \Delta_1 \mathcal{P}^\perp (\mathcal{P}^\perp (\Delta_1 - \lambda) \mathcal{P}^\perp)^{-1} \mathcal{P}^\perp \Delta_1) \mathcal{P}) \upharpoonright_{\ell^2(\{0\})}. \quad (5.6)$$

In particular, if the operator $\Delta_1 - \lambda \mathcal{P}^\perp$ is invertible, then the expression (5.6) can be written in the form

$$Q^{-1}(\lambda) = (\mathcal{P}(\Delta_1 - \lambda \mathcal{P}^\perp)^{-1} \mathcal{P}) \upharpoonright_{\ell^2(\{0\})}. \quad (5.7)$$

Proof. Let $\lambda \in \mathbb{R}$ not be an eigenvalue of the operator $\mathcal{P} \Delta_1 \mathcal{P}$. For each $\vartheta \in \mathbb{T}$, using (4.28), (4.29) and (5.2), we obtain the Feshbach map for the operator $\Delta(\vartheta) - \lambda$:

$$\begin{aligned} \mathcal{F}(\vartheta, \lambda) &= P(\Delta_0(\vartheta) - \lambda)P + Q(\lambda) = P(h - \mu)P + Q(\lambda), \\ \mu &= \lambda - (2 - 2 \cos \vartheta), \end{aligned} \quad (5.8)$$

where P is the orthogonal projection of $\ell^2(V^c)$ onto the subspace $\ell^2(\mathbb{Z})$, h is the Laplacian on \mathbb{Z} , the potential $Q(\lambda)$ has the support $\{0\}$ and is given by (5.6) or, in the case when $\Delta_1 - \lambda \mathcal{P}^\perp$ is invertible, by (5.7). Then, due to Proposition 4.3 and the identity (5.8), we have

$$\lambda \in \sigma_p(\Delta(\vartheta)) \Leftrightarrow 0 \in \sigma_p(\mathcal{F}(\vartheta, \lambda)) \Leftrightarrow \mu \in \sigma_p(h + Q(\lambda)). \quad (5.9)$$

It is well known that the Schrödinger operator $h + Q$ on \mathbb{Z} with a potential Q having a support at a single point has the unique eigenvalue

$$\mu = \begin{cases} 2 + \sqrt{4 + Q^2}, & Q > 0, \\ 2 - \sqrt{4 + Q^2}, & Q < 0. \end{cases} \quad (5.10)$$

Combining this with (5.9) and using the second identity in (5.8) and the fact that any eigenvalue λ of $\Delta(\vartheta)$ satisfies $\lambda \geq 2 - 2 \cos \vartheta$, we obtain

$$\sigma_p(\Delta(\vartheta)) = \{\lambda \in \mathbb{R} : \lambda - \sqrt{4 + Q^2(\lambda)} = 4 - 2 \cos \vartheta, \quad Q(\lambda) > 0\}, \quad (5.11)$$

which, due to the definition (2.12) of the guided spectrum, yields (5.4) and $\lambda \notin \mathfrak{s}_{fb}(\Delta)$.

Let $\lambda \in \mathbb{R}$ be an eigenvalue of the operator $\mathcal{P} \Delta_1 \mathcal{P}$. Then, due to Proposition 2.2, $\{\lambda\}$ is a guided flat band of the Laplacian Δ on Γ . Thus, $\mathfrak{s}_{fb}(\Delta)$ has the form (5.5). ■

Remark. If $Q(\lambda) < 0$ for some $\lambda > 8$, then λ lies in the gap of the spectrum of the Laplacian Δ on the perturbed square lattice.

5.1. Lattice with p pendant edges at each vertex of $\mathbb{Z} \times \{0\}$. Let $\Gamma_1 = (V_1, \mathcal{E}_1)$ be a star graph of order $p + 1$, i.e., a tree on $p + 1$ vertices v_1, v_2, \dots, v_{p+1} with one vertex $v_1 = 0$ having degree p and the other p having degree 1. We assume that

$$V_{01} = \mathbb{Z} \cap V_1 = \{0\} \quad (5.12)$$

and consider the perturbed square lattice $\Gamma = \mathbb{L}^2 \cup \Gamma_1^g$, see Fig.6.

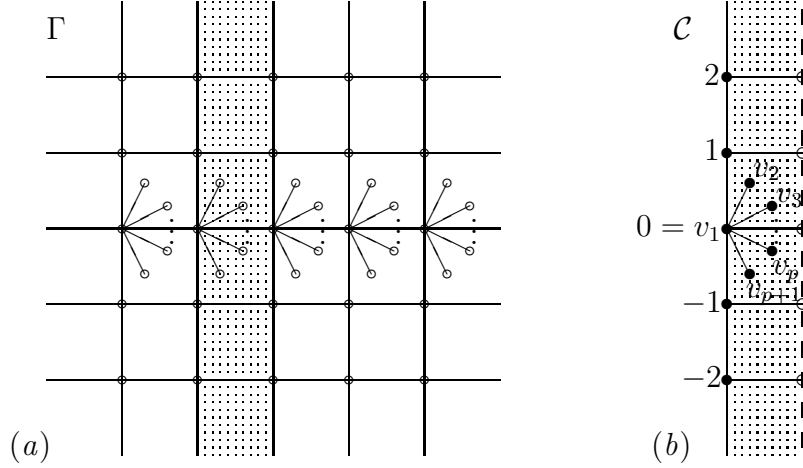


FIGURE 6. a) The perturbed square lattice $\Gamma = \mathbb{L}^2 \cup \Gamma_1^g$; b) the perturbed cylinder $\mathcal{C} = \Gamma/\mathbb{Z}$.

Proposition 5.2. *The guided spectrum of the Laplacian Δ on the perturbed square lattice $\Gamma = \mathbb{L}^2 \cup \Gamma_1^g$, where Γ_1 is a star graph of order $p + 1$ satisfying (5.12), has the form*

$$\begin{aligned} \mathfrak{s}(\Delta) &= \mathfrak{s}_{ac}(\Delta) \cup \mathfrak{s}_{fb}(\Delta), & \mathfrak{s}_{ac}(\Delta) &= \mathfrak{s}_1(\Delta), \\ \mathfrak{s}_{fb}(\Delta) &= \mathfrak{s}_2(\Delta) \cup \dots \cup \mathfrak{s}_p(\Delta) = \{1\}, \end{aligned}$$

i.e., the guided flat band $\{1\}$ has the multiplicity $p - 1$ and

$$\mathfrak{s}_1(\Delta) \subset [p + 3, p + 8], \quad |\mathfrak{s}_1(\Delta)| < 4.$$

Moreover, $\lim_{p \rightarrow \infty} |\mathfrak{s}_1(\Delta)| = 2\beta_+ = 4$, where β_+ is defined in (2.18).

Remark. For example, for $p = 1$ and $p = 2$ the guided band $\mathfrak{s}_1(\Delta)$ has the form

$$\mathfrak{s}_1(\Delta) \approx [4, 38; 8, 30] \quad \text{and} \quad \mathfrak{s}_1(\Delta) \approx [5, 18; 9, 01],$$

respectively.

Proof. The $(p + 1) \times (p + 1)$ matrix $\Delta_1 = \{\Delta_{uv}^1\}_{u,v \in V_1}$ defined by (4.17) for the Laplacian Δ_1 on the star-graph Γ_1 has the form

$$\Delta_1 = \begin{pmatrix} p & b \\ b^* & I_p \end{pmatrix}, \quad b = (-1, \dots, -1) \in \mathbb{C}^p, \quad (5.13)$$

where I_p is the identity $p \times p$ matrix. Since $\lambda = 1$ is an eigenvalue of the matrix Δ_1 of multiplicity $p - 1$, due to Proposition 2.2, the Laplacian Δ on Γ has a guided flat band $\{1\}$ of multiplicity $p - 1$.

Now we find the absolutely continuous guided spectrum $\mathfrak{s}_{ac}(\Delta)$. Since the number of guided bands $N \leq p$ and the guided flat band $\{1\}$ has multiplicity $p - 1$, the absolutely continuous guided spectrum $\mathfrak{s}_{ac}(\Delta)$ consists of at most one band $\mathfrak{s}_1(\Delta) = [\lambda_1^-, \lambda_1^+]$ and, due to Proposition 5.1, has the form (5.4), where $Q(\lambda) = \frac{p\lambda}{\lambda-1}$. The inequality $Q(\lambda) > 0$ gives that $\lambda > 1$. The graphs of the function $f(\lambda) = \sqrt{4 + Q^2(\lambda)}$ for $\lambda > 1$ and of the functions $g_1(\lambda) = \lambda - 2$, $g_2(\lambda) = \lambda - 6$ are shown in Fig.7. Thus, the absolutely continuous guided spectrum $\mathfrak{s}_{ac}(\Delta)$ consists of exactly one guided band $\mathfrak{s}_1(\Delta) = [\lambda_1^-, \lambda_1^+]$ and $|\mathfrak{s}_1(\Delta)| = \lambda_1^+ - \lambda_1^- < 4$, see Fig.7.

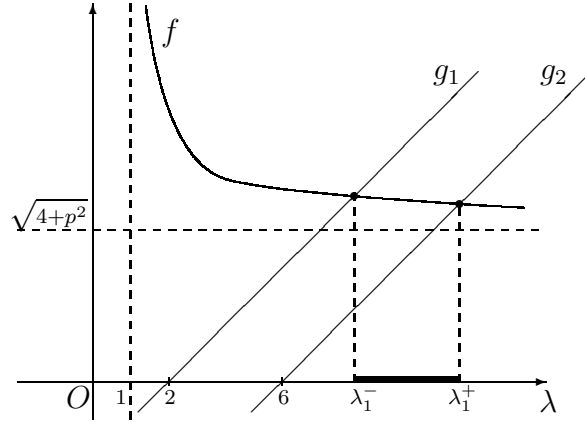


FIGURE 7. The graphs of the functions f , g_1 , g_2 .

From the identity $f(\lambda_1^-) = g_1(\lambda_1^-)$ we obtain that λ_1^- is a root of the equation

$$h(\lambda) \equiv \lambda^3 - 6\lambda^2 + (9 - p^2)\lambda - 4 = 0. \quad (5.14)$$

Using that

$$h(p+3) = -4 < 0, \quad h(p+4) = 2p^2 + 9p > 0,$$

we obtain that the equation (5.14) has a unique solution

$$\lambda_1^- \in [p+3, p+4]. \quad (5.15)$$

This and $|\mathfrak{s}_1(\Delta)| < 4$ give that $\mathfrak{s}_1(\Delta) \subset [p+3, p+8]$.

We have $\lambda_1^\pm \rightarrow \infty$ as $p \rightarrow \infty$ and $f(\lambda) = \sqrt{4 + p^2} + o(1)$ as $\lambda \rightarrow \infty$. Then

$$\lambda_1^- \rightarrow 2 + \sqrt{4 + p^2}, \quad \lambda_1^+ \rightarrow 6 + \sqrt{4 + p^2},$$

which yields that $|\mathfrak{s}_1(\Delta)| \rightarrow 4$ as $p \rightarrow \infty$. On the other hand, the number $\beta_+ = 2$, since for each vertex of the cylinder \mathcal{C} there are two bridges starting at this vertex. Thus, we have $|\mathfrak{s}_1(\Delta)| = 2\beta_+ + o(1)$ as $p \rightarrow \infty$. ■

5.2. Lattice with mandarin guides. An *s-mandarin graph* is a graph consisting of two vertices and s edges connecting these vertices. Let $\Gamma_1 = (V_1, \mathcal{E}_1)$ be a union of two *s-mandarin* graphs with one common vertex $v_1 = 0$, see Fig.2.b. We assume that

$$V_{01} = \mathbb{Z} \cap V_1 = \{0\} \quad (5.16)$$

and consider the perturbed square lattice $\Gamma = \mathbb{L}^2 \cup \Gamma_1^g$, see Fig.2.

Proposition 5.3. *The guided spectrum of the Laplacian Δ on the perturbed square lattice $\Gamma = \mathbb{L}^2 \cup \Gamma_1^g$, where Γ_1 is a union of two s -mandarin graphs with one common vertex satisfying (5.16), has the form*

$$\mathfrak{s}(\Delta) = \mathfrak{s}_{ac}(\Delta) \cup \mathfrak{s}_{fb}(\Delta), \quad \mathfrak{s}_{ac}(\Delta) = \mathfrak{s}_1(\Delta), \quad \mathfrak{s}_{fb}(\Delta) = \mathfrak{s}_2(\Delta) = \{s\},$$

where

$$\mathfrak{s}_1(\Delta) \subset [3s + 1, 3s + 6] \quad \text{for } s \geq 2, \quad |\mathfrak{s}_1(\Delta)| < 4.$$

Moreover, $\lim_{s \rightarrow \infty} |\mathfrak{s}_1(\Delta)| = 4$.

Remarks. 1) For example, for $s = 1$ and $s = 2$ the guided band $\mathfrak{s}_1(\Delta)$ has the form

$$\mathfrak{s}_1(\Delta) \approx [5, 18; 9, 01] \quad \text{and} \quad \mathfrak{s}_1(\Delta) \approx [7, 75; 11, 26],$$

respectively.

2) For $s \leq 8$ the guided flat band $\{s\}$ is embedded into the absolutely continuous spectrum of the Laplacian Δ . For $s > 8$ it lies in a gap.

Proof. The matrix $\Delta_1 = \{\Delta_{uv}^1\}_{u,v \in V_1}$ defined by (4.17) for the Laplacian Δ_1 on the graph Γ_1 has the form

$$\Delta_1 = s \begin{pmatrix} 2 & b \\ b^* & I_2 \end{pmatrix}, \quad b = (-1, -1). \quad (5.17)$$

Since $\lambda = s$ is a simple eigenvalue of the matrix Δ_1 , due to Proposition 2.2, the Laplacian Δ on Γ has a guided flat band $\{s\}$.

The absolutely continuous guided spectrum $\mathfrak{s}_{ac}(\Delta)$ consists of at most one band $\mathfrak{s}_1(\Delta) = [\lambda_1^-, \lambda_1^+]$ and, due to Proposition 5.1, has the form (5.4), where $Q(\lambda) = \frac{2s\lambda}{\lambda-s}$. The inequality $Q(\lambda) > 0$ gives that $\lambda > s$. The graphs of the function $f(\lambda) = \sqrt{4 + Q^2(\lambda)}$ for $\lambda > s$ and of the functions $g_1(\lambda) = \lambda - 2$, $g_2(\lambda) = \lambda - 6$ are shown in Fig.8. Thus, the absolutely continuous guided spectrum $\mathfrak{s}_{ac}(\Delta)$ consists of exactly one guided band $\mathfrak{s}_1(\Delta) = [\lambda_1^-, \lambda_1^+]$ and $|\mathfrak{s}_1(\Delta)| = \lambda_1^+ - \lambda_1^- < 4$, see Fig.8.

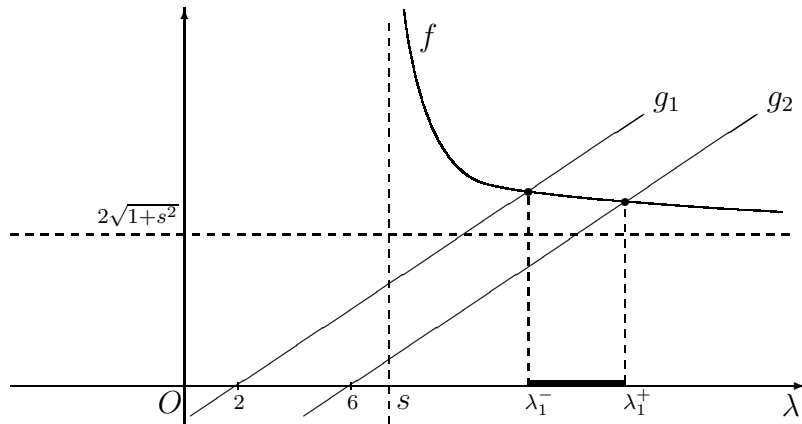


FIGURE 8. The graphs of the functions f , g_1 , g_2 .

From the identity $f(\lambda_1^-) = g_1(\lambda_1^-)$ we obtain that λ_1^- is a root of the equation

$$h(\lambda) \equiv (\lambda - 4)(\lambda - s)^2 - 4s^2\lambda = 0. \quad (5.18)$$

Using that

$$h(3s + 1) = -4s^2 - 9s - 3 < 0, \quad h(3s + 2) = 4(2s^2 - s - 2) > 0 \quad \text{for } s \geq 2,$$

we obtain that for $s \geq 2$ the equation (5.18) has a unique solution

$$\lambda_1^- \in [3s + 1, 3s + 2]. \quad (5.19)$$

This and $|\mathfrak{s}_1(\Delta)| < 4$ give that $\mathfrak{s}_1(\Delta) \subset [3s + 1, 3s + 6]$.

We have $\lambda_1^\pm \rightarrow \infty$ as $s \rightarrow \infty$ and $f(\lambda) = 2\sqrt{1 + s^2} + o(1)$ as $\lambda \rightarrow \infty$. Then

$$\lambda_1^- \rightarrow 2 + 2\sqrt{1 + s^2}, \quad \lambda_1^+ \rightarrow 6 + 2\sqrt{1 + s^2},$$

which yields that $|\mathfrak{s}_1(\Delta)| \rightarrow 4$ as $s \rightarrow \infty$. ■

5.3. Lattice with path guides. Let $\Gamma_1 = (V_1, \mathcal{E}_1)$ be a path of length 2, i.e., a connected graph with two vertices v_1 and v_3 having degree 1 and a vertex v_2 having degree 2 and let $\Gamma_t = (V_1, \mathcal{E}_t)$ be a finite graph obtained from the graph Γ_1 considering each edge of Γ_1 to have the multiplicity t . We assume that $V_{01} = \mathbb{Z} \cap V_1 = \{v_1 = 0\}$ and consider the perturbed square lattice $\Gamma = \mathbb{L}^2 \cup \Gamma_t^g$, see Fig.9.

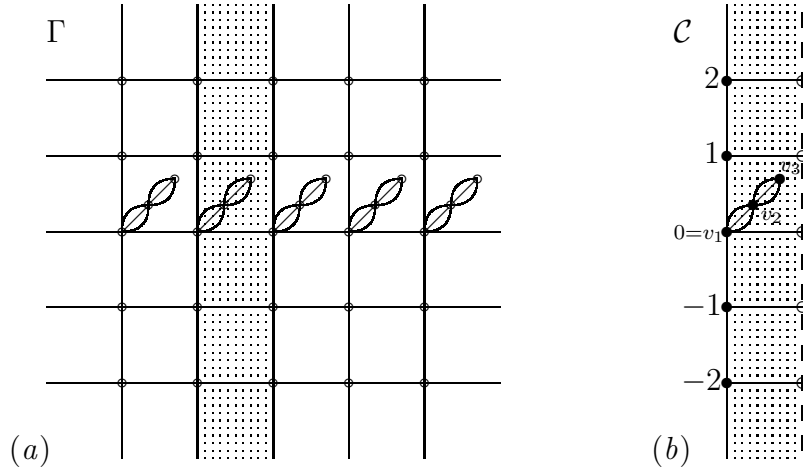


FIGURE 9. a) The perturbed square lattice $\Gamma = \mathbb{L}^2 \cup \Gamma_t^g$; b) the perturbed cylinder $\mathcal{C} = \Gamma/\mathbb{Z}$.

Proposition 5.4. *The guided spectrum of the Laplacian Δ on the perturbed square lattice $\Gamma = \mathbb{L}^2 \cup \Gamma_t^g$ has the form*

$$\mathfrak{s}(\Delta) = \mathfrak{s}_{ac}(\Delta) = \begin{cases} \mathfrak{s}_1(\Delta), & \text{if } t = 1, 2 \\ \mathfrak{s}_1(\Delta) \cup \mathfrak{s}_2(\Delta), & \text{if } t \geq 3 \end{cases},$$

where $\mathfrak{s}_2(\Delta) \subset [t, 2t]$ and $\mathfrak{s}_1(\Delta) = [\lambda_1^-, \lambda_1^+]$,

$$\lambda_1^- = 2 + \sqrt{4 + t^2} + o(1), \quad \lambda_1^+ = 6 + \sqrt{4 + t^2} + o(1), \quad \text{as } t \rightarrow \infty. \quad (5.20)$$

Proof. The matrix $\Delta_1 = \{\Delta_{uv}^1\}_{u,v \in V_1}$ defined by (4.17) for the Laplacian Δ_1 on the graph Γ_t has the form

$$\Delta_1 = t \begin{pmatrix} 1 & b \\ b^* & \Delta_D \end{pmatrix}, \quad b = (-1, 0), \quad \Delta_D = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}. \quad (5.21)$$

Since the matrices Δ_1 and Δ_D have no equal eigenvalues, due to Proposition 5.1, the Laplacian Δ on Γ has no guided flat bands. Then the absolutely continuous guided spectrum $\mathfrak{s}_{ac}(\Delta)$ consists of at most two bands and, due to Proposition 5.1, has the form (5.4), where

$$Q(\lambda) = \frac{t\lambda(\lambda - 2t)}{\lambda^2 - 3t\lambda + t^2}.$$

The inequality $Q(\lambda) > 0$ gives that

$$\lambda \in \left(\frac{3-\sqrt{5}}{2}t, 2t\right) \cup \left(\frac{3+\sqrt{5}}{2}t, +\infty\right). \quad (5.22)$$

For $t \geq 4$ the graphs of the function $f = \sqrt{4 + Q^2(\lambda)}$ for $\lambda > \frac{3-\sqrt{5}}{2}t$ and of the functions $g_1(\lambda) = \lambda - 2$, $g_2(\lambda) = \lambda - 6$ are shown in Fig.10. Thus, for $t \geq 4$ the absolutely continuous guided spectrum $\mathfrak{s}_{ac}(\Delta)$ consists of exactly two guided bands $\mathfrak{s}_s(\Delta) = [\lambda_s^-, \lambda_s^+]$, $s = 1, 2$. Since the Laplacian Δ_1 on the graph Γ_t has the positive eigenvalues $\zeta_1 = 3t$ and $\zeta_2 = t$, Corollary 4.1 and the formula (5.22) give that $\mathfrak{s}_2(\Delta) \subset [t, 2t]$.

We have $\lambda_1^\pm \rightarrow \infty$ as $t \rightarrow \infty$ and $f(\lambda) = \sqrt{4 + t^2} + o(1)$ as $\lambda \rightarrow \infty$. Then we get (5.20).

From (5.4) by a direct calculation we obtain the absolutely continuous guided spectrum of the Laplacian Δ for $t \leq 3$:

$$\mathfrak{s}_{ac}(\Delta) = \begin{cases} \mathfrak{s}_1(\Delta) \approx [4, 47; 8, 31], & t = 1 \\ \mathfrak{s}_1(\Delta) \approx [6, 66; 9, 45], & t = 2 \\ \mathfrak{s}_1(\Delta) \cup \mathfrak{s}_2(\Delta) \approx [4, 62; 6] \cup [9, 51; 11, 44], & t = 3 \end{cases}.$$

■

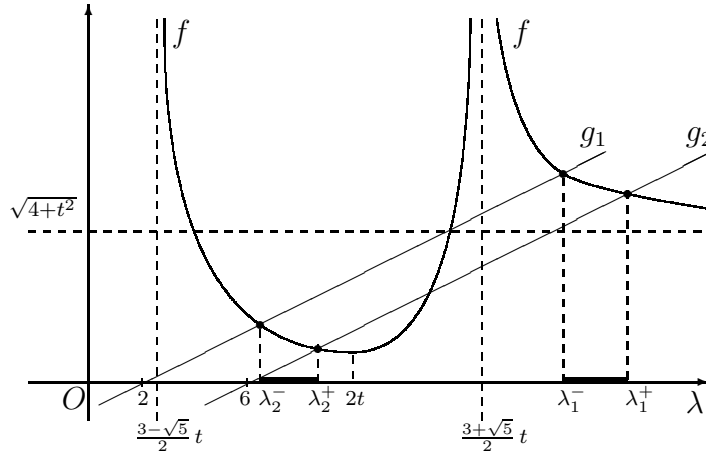


FIGURE 10. The graphs of the functions f , g_1 , g_2 ; $t \geq 4$.

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